ON A HYPERBOLIC WAVE EQUATION IN UNBOUNDED DOMAINS

by

Eleftherios Stratikopoulos

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Supervisor: P. G. Papadopoulos

University of West Attica
Faculty of Engineering
Department of Electrical and Electronics Engineering

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ΣΤΟΙΧΕΙΑ ΦΟΙΤΗΤΗ

Ονομ/νυμο: Ελευθέριος Στρατικόπουλος

Αριθμός Μητρώου: 50347568

ΕΠΙΒΛΕΠΟΝ ΜΕΛΟΣ ΔΕΠ

Περικλής Γ. Παπαδόπουλος

Καθηγητής ΠΑΔΑ

Πανεπιστήμιο Δυτικής Αττικής
Σχολή Μηχανικών
Τμήμα Ηλεκτρολόγων και Ηλεκτρονικών Μηχανικών
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Παράβαση της ανωτέρω ακαδημαϊκής μου ευθύνης αποτελεί ουσιώδη λόγο για την ανάκληση του διπλώματός μου.»

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Eleftherios Stratikopoulos
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……to my family

Abstract

The aim of this thesis is the study of the quasilinear damped wave equation of Kirchhoff's type with variable diffusion coefficient in all of $\mathbb{R}^N$. For the functional analysis of the time dependent problem, we make use of the homogeneous Sobolev spaces and of the generalized Sobolev embeddings, followed by the preceding studies of the Kirchhoff's type problem. In strong connection with the corresponding natural phenomena, we obtain results concerning the local (unique) existence of the solutions using the Faedo-Galerkin approximation and the Banach Fixed-Point Theorem. We also prove the global existence and energy estimates of the solutions using the method of the modified potential well. We complete our study with the blow-up analysis of the solutions for initial data of negative energy using the concavity method, where for the discrete case (a=2) we prove the modification of the upper-bound of the time T.

Keywords: Infinite Dimensional Dynamical Systems, Quasilinear Hyperbolic Wave Equations, Nonlinear Problems, Dissipation, Semigroup Theory, Banach Fixed-Point Theorem, Galerkin Method, Faedo-Galerkin Approximation, Unbounded Domains, Homogeneous Sobolev Spaces, Weighted Lebesgue-Sobolev Spaces, Generalized Poincare Inequality, Generalized Sobolev Embeddings, Blow-Up, Concavity Method, Potential Well, Modified Potential Well, Kirchhoff's Strings.

Notation. We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for simplicity we use the symbols L^p, $1 \leq p \leq \infty$, $\mathcal{D}^{1.2}$, respectively, for the spaces $L^p(\mathbb{R}^N)$, $\mathcal{D}^{1.2}(\mathbb{R}^N)$, respectively; $\| \cdot \|_p$ for the norm $\| \cdot \|_{L^p(\mathbb{R}^N)}$. By $\mathcal{L}(V, W)$ or sometimes by $\mathscr{L}$(V,W) we denote the space of linear operators from V to W. Also, sometimes differentiation with respect to time is denoted by a dot over the function. Furthermore, we have used the notation $\mathfrak{R}$ and $\mathfrak{R}^N$ for the spaces $\mathbb{R}$ and $\mathbb{R}^N$, respectively. All the constants are considered in a generic sense. The end of the proofs is denoted by Q.E.D (*quod erat demonstrandum* = which had to be shown) and the end of a theorem or lemma whose proof is not given, is denoted by ■.

CONTENTS.

CHAPTER II.

Existence of global solutions.

Catalog of Images

A. Preliminary

1. Functional Analysis

In mathematics, the word "function" is being handled as a map from one space into another or as an operator that acts on the elements of some properly defined space. The study of the physical phenomena that rule the known limits of our world invokes primarily the study of the space in which these phenomena are domiciled. As any mathematical problem of several variables becomes properly a problem in vector calculus the mentioned spaces are called by thy name "vector spaces" or "linear spaces", and in this chapter, we will delve into the aspects of the most known vector spaces that are used frequently in this work and in applied mathematics generally.

1.1 Banach Spaces

In our notation throughout this section, we denote by V a real linear space[1].

Definition 1.1.1. *Given a linear space V, a mapping* $\|\ \ \|: V \to [0, \infty)$ *is called a norm if the following properties are satisfied.*

(i) $\|u\| \geq 0$ *for any* $u \in V$, *and* $\|u\| = 0$ *if and only if* $u = 0$,
(ii) $\|au\| = |a|\|u\|$ *for any* $u \in V$ *and* $a \in \mathfrak{S}$,
(iii) $\|u + v\| \leq \|u\| + \|v\|$ *for any* $u, v \in V$, *known as the triangle inequality*.

The symbol $\mathfrak{S}$ in property (ii) denotes the scalar field ($\mathfrak{R}$ or $\mathfrak{C}$).

Hereafter we assume V is a normed linear space, and we adapt the notation (V, $\|\ \ \|$).

Remark. A norm induces a metric, i.e., a way of taking the distance between two elements of the space. So we regard every normed space as a metric space, in which the distance $d(u, v)$ between u and v is $\|u - v\|$. The properties of the distance are

(i) $0 \leq d(u, v) < \infty$ for all u and υ,
(ii) $d(u, v) = 0$ *if and only if* $u = v$,
(iii) $d(u, v) = d(v, u)$ *for all u and υ*,
(iv) $d(u, v) \leq d(u, w) + d(w, v)$ *for all u, υ, w*.

1. See in References [4].

Definition 1.1.2. *We say a sequence* $\{u_k\}_{k=1}^{\infty} \subset V$ *converges to* $u \in V$, *written*

$$u_k \to u,$$

if

$$\lim_{k\to\infty} \|u_k - n\| = 0.$$

Definition 1.1.3. (a) *A sequence* $\{u_k\}_{k=1}^{\infty} \subset$ V is called *a Cauchy sequence if for every* ε > 0 *there exists* N > 0 *such that*

$$\|u_k - u_l\| < \varepsilon \text{ for all } k, l \geq N.$$

(b) *A normed space V is said to be complete if every Cauchy sequence from the space converges to an element in the space, i.e., if* $\{u_k\}_{k=1}^{\infty} \subset$ V *is a Cauchy sequence, and there exists* $u \in V$ *such that* $\{u_k\}_{k=1}^{\infty}$ *converging to u then the space V is complete.*
(c) *A complete normed space is called a* ***Banach space***.

Definition 1.1.4. (a) *Let K be a subset of a normed vector space V. We say K is compact if, for every open covering of K, there is a finite subcover that also covers K. This is equivalently regarded as K having the Borel-Heine property.*
(b) *Equivalently, K is compact if every sequence* $\{u_k\}_{k=1}^{\infty} \subset K$ *contains a convergent subsequence which converges to an element* $u \in V$.

Definition 1.1.5. *Let* $K_1 \subset K_2$ *be two subsets in a normed space V. We say the set* K_1 *is dense in* K_2 *if for any* $u \in K_2$ *and any* $\varepsilon > 0$, *there is a* $v \in K_1$ *such that* $d(u, v) < \varepsilon$.

Remark. The geometrical interpretation of the Definition 1.1.5 is that we can select any positive number which will denote the accuracy of closeness (density) between two subsets of a normed vector space.

Definition 1.1.6. *A normed vector space is called separable if it contains a countable dense subset.*

1.2 Hilbert Spaces

Definition 1.2.1. *Given a real linear space V, a mapping* $(.,.): V \times V \to [0, \infty)$ *is called an inner product if the following properties are satisfied.*

(i) $(u,u) \geq 0$ *and* $(u,u) = 0$ *if and only if* $u = 0$; *for any* $u \in V$,
(ii) $(u,v) = (v,u)$ *for any* $u \in V$,
(iii) $(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w)$, *for any* $u, v, w \in V$ *and* $\alpha, \beta \in \mathfrak{R}$.

The space V equipped with the inner product $(.,.)$ *is called an inner product space.*

An inner product $(.,.)$ induces a norm through the formula

$$\|u\| := \sqrt{(u,u)},\, u \in V.$$

Remark. In verifying the triangle inequality for the above quantity, we need to use the Schwarz inequality; $|(u,v)| \leq \sqrt{(u,u)(v,v)}\ \forall u,v \in V.$

Definition 1.2.2. *A complete inner product space is called a* ***Hilbert space****. This means that an inner product space V is a Hilbert space if V is a Banach space under the norm induced by the inner product.*

1.3 L^p and Sobolev Spaces

In introductory calculus the integrability over a properly defined region deals with the values of the integrable function over a volume. The physical correspondence and in a matter of fact the information behind the function in the region of interest denotes a finite value necessity as infinity is hiding more than itself.
This means that if $\Omega \subset \Re^N$ is a non-empty open set of N-dimensional Euclidean space, the integration over Ω of a properly defined measurable function, $f : \Omega \rightarrow \Re$, needs to be finite, i.e.,

$$\int_\Omega f(x)\,dx < \infty.$$

Having set the characteristics of our function we are able to construct a space for these functions with the above property.

Definition 1.3.1. *Let* $\Omega \subset \Re^N$ *be a non-empty open set. For* $p \in [1,\infty)$*, we define the space of all Lebesgue integrable functions as follows*

$$L^p(\Omega) := \left\{u : \Omega \rightarrow \mathbb{R} \middle| \int_\Omega |u(x)|^p\,dx < \infty\right\}.$$

Proposition. (a) The L^p spaces are Banach spaces under the norm

$$\|u\|_{L^p(\Omega)} := \left[\int_\Omega |u(x)|^p\,dx\right]^{1/p} < \infty,$$

and the space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions equipped with the norm

$$\|u\|_{L^\infty(\Omega)} := inf\ sup|u(x)| < \infty.$$

(b) In the case $p = 2$, L^p spaces are also Hilbert spaces with the inner product

$$(u,v) := \int_{\Omega} u(x)v(x)\,dx. \blacksquare$$

Of equivalent importance are the so called *weighted-Lebesgue* spaces or the L_g^p spaces as we usually refer to; the definition of those is given below.

Definition 1.3.2. *Let g be a positive continuous function on Ω, called a weight-function. We can define weighted spaces $L_g^p(\Omega)$ as follows*

$$L_g^p(\Omega) := \left\{u\ measurable \middle| \int_\Omega g(x)|u(x)|^p\,dx < \infty\right\}, \textit{for } 1 \le p < \infty$$

$$L_g^\infty(\Omega) := \{u\ measurable | ess \sup g(x)|u(x)| < \infty\}.$$

Proposition. The L_g^p spaces are Banach spaces with the norms

$$\|u\|_{p,g} := \left\{\int_\Omega g(x)|u(x)|^p\,dx\right\}^{1/p} \textit{ for } 1 \le p < \infty$$

$$\|u\|_{\infty,g} := ess \sup g(x)|u(x)| \quad \textit{for } x \in \Omega. \blacksquare$$

Before we procced with the definitions of the *Sobolev spaces*, firstly we will develop briefly the background that brought them out.

In the study of PDEs an "operator" treatment, i.e., the recasting of a partial differential equation as an operator acting on appropriate linear spaces with the operator encoding the structure of the PDE, postulates a specific treatment in the choice of the proper spaces. The honorable mathematician *Sergei Lvovich Sobolev* constructed the spaces (which took their name after him) and set the proper framework of the "operator" treatment.

Definition 1.3.3. *Let k be a nonnegative integer and $p \in [1,\infty]$. The Sobolev space $W^{k,p}(\Omega)$ is defined as follows*

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) | \vartheta^{a}u \in L^p(\Omega), for\ each\ |a| \le k\},$$

where a *denotes a multi-index, i.e., an ordered collection of N non-negative integers, a = (a₁, a₂, . . ., a_N), the quantity* $|a| = \sum_{i=1}^{N} a_i$ *is said to be the length of a, and the expression $\theta^{a}u$ denotes the* ath *weak derivative of u*[2].

Proposition. (a) The norm in the space $W^{k,p}(\Omega)$ is defined as

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left[\sum_{|a|\le k} \|\vartheta^{a}u\|_{L^p(\Omega)}^p\right]^{1/p}, & 1 \le p < \infty \\ max_{|a|\le k}\|\vartheta^{a}u\|_{L^\infty(\Omega)}, & p = \infty. \end{cases}$$

2. For the definition of *weak derivatives*, see in Chapter 7 of [4], or in Chapter 5 of [5].

(b) When $p = 2$, we use the notation $H^k(\Omega)$ for the *Sobolev spaces* $W^{k,2}(\Omega)$. ■

The following theorem provides us with the concept of completeness of *Sobolev spaces* (for the proof see Chapter 7 of [4]).

Theorem 1.3.4. *The Sobolev space* $W^{k,p}(\Omega)$ *is a Banach space.* ■

Before we procced in the concepts of linear operators we give another property of major importance for the *Sobolev space* $H^k(\Omega)$.

Corollary 1.3.5. The *Sobolev space* $H^k(\Omega)$ *is a Hilbert space with the inner product*

$$(u,v)_k \coloneqq \int_\Omega \sum_{|a|\le k} \vartheta^a u(x)\, \vartheta^a v(x)\, dx, \quad u,v \in H^k(\Omega).$$

More about the *Sobolev spaces* we will see in Chapter II of this work.

1.4 Bounded and compact linear operators

In the previous section we referred to "operators" as mappings from one space into another that enclose the structure of a PDE. The concept of operator is important in our study and is worth little of our time to review some of their basic properties.
Let us assume that X, and Y are two sets and $\mathfrak{T}$ an operator from X to Y. This means that $\mathfrak{T}$ is a function which assigns to each element in a subset of X a unique element in Y.

Definition 1.4.1. (a) The *domain* $\mathcal{D}(\mathfrak{T})$ of $\mathfrak{T}$ is the set of all elements of X in where $\mathfrak{T}$ is properly defined, i.e.,

$$\mathcal{D}(\mathfrak{T}) = \{u \in X \mid \mathfrak{T}(u) \text{ is defined}\}.$$

(b) The *range* $\mathcal{R}(\mathfrak{T})$ of $\mathfrak{T}$ is the set of all the elements in Y generated by $\mathfrak{T}$, i.e.,

$$\mathcal{R}(\mathfrak{T}) = \{v \in Y \mid v = \mathfrak{T}(u) \text{ for some } u \in \mathcal{D}(\mathfrak{T})\}.$$

(c) The *null set* $\mathcal{N}(\mathfrak{T})$ of $\mathfrak{T}$ is defined as the set of elements of X in where $\mathfrak{T}$ is mapping to the zero element, i.e.,

$$\mathcal{N}(\mathfrak{T}) = \{u \in X \mid \mathfrak{T}(u) = 0\}.$$

Definition 1.4.2. An operator from X to Y is called *bijective* if it is injective (one-to-one) and surjective, i.e., if $u_1 \neq u_2 \rightarrow \mathfrak{T}(u_1) \neq \mathfrak{T}(u_2)$ and $\mathcal{R}(\mathfrak{T}) = Y$, respectively.

Definition 1.4.3. *An operator* $\mathfrak{T}$: X → Y is called *linear* if

$$\mathfrak{T}(\lambda u_1 + \mu u_2) = \lambda\,\mathfrak{T}(u_1) + \mu\,\mathfrak{T}(u_2)$$

for all $u_1, u_2 \in X$, $\lambda, \mu \in \mathfrak{R}$.

Proposition. A linear operator $\mathfrak{T}$: X → Y is *bounded* (or *continuous* since for a linear operator boundedness ≡ continuity) *if and only if* there exists a constant $c \geq 0$ such that

$$\|\mathfrak{T}u\|_Y \leq c\,\|u\|_X \quad \forall\, u \in X.$$

For the proof of the above inequality see Section 2.2 (Continuous linear operators) of [4]. ■

Throughout this work we will use the notation $\mathscr{L}$(X, Y) for the set of all the continuous (or bounded) linear operators from X to Y, and since this set is a *linear space* induces a norm over the space.

Definition 1.4.4. *If* $\mathfrak{T} \in \mathscr{L}$(X, Y) *then the operator norm of* $\mathfrak{T}$ is given by

$$\|\mathfrak{T}\|_{X,Y} = \sup\,\{\|\mathfrak{T}u\|_Y / \|u\|_X\} \; for \;\; 0 \neq u \in X$$

and having the following *compatibility* property

$$\|\mathfrak{T}u\|_Y \leq \|\mathfrak{T}\|_{X,Y}\,\|u\|_X \quad \forall\, u \in X.$$

Before we give the definition for the *compactness* of a linear operator, we will emphasize in the set $\mathscr{L}$(X, Y).

When dealing with mappings over normed vector spaces the concept of *approximation* induces that of *convergence*, i.e., the question about if a normed vector space is *complete* under the equipped norm.

Someone obviously may wonder why do we need approximations, but before going there, let us assume that we have an equation of the following form

$$\mathfrak{T}u = v,$$

where $\mathfrak{T}$: X → Y, $u \in X$ and $v \in Y$.

The operator may be an integral operator (so the above is an integral-equation), a differential operator (and so the above is a differential equation) or a combination of those.

In many cases the difficulty with proceeding with the analysis of a given problem that takes the above form, has to do with the operator itself. In such cases the integral or differential operator is often approximated by a sequence of operators of a simpler form.

Since we have explained the necessity of approximations, we may return to the question of whether the set $\mathscr{L}$ (X, Y) is a complete space.

Theorem 1.4.5. Let X be a normed space, and Y a Banach space. Then, the set $\mathscr{L}$ (X, Y) is a Banach space (the proof can be found in Section 2.2.1 ($\mathscr{L}$ (X, Y) as a Banach space) in [4]).

We will end this introduction on Functional analysis with the definition of compactness of an operator.

Definition 1.4.6. *An operator $A \epsilon$* $\mathscr{L}$ (X, Y) is called *compact* if for each *bounded sequenc*e $\{u_k\}_{k=1}^{\infty} \subset$ X, the *sequence* $\{Au_k\}_{k=1}^{\infty}$ is *pre-compact* in Y, i.e., *if there exists a subsequence* $\left\{u_{k_j}\right\}_{j=1}^{\infty}$ such that $\left\{Au_{k_j}\right\}_{j=1}^{\infty}$ *converges* in Y.

2. Embeddings and inequalities

In the study of PDEs the framework of operator analysis induces, as we saw, the study and selection of the proper spaces on where the operator acts. According to our problem and the characteristics of our operator, the existence of embeddings is rising from the fact that some properties of the operator may be valid in larger spaces. Having so a relation between more treatable spaces with the spaces that define the framework of our problem, and using the *extension theorems* to develop an *embedding-chain* among them we are able to continue with the study of our problem (see more in Chapter II).

2.1 Definitions

Definition 2.1.1. We say a space X is *embedded* in Y and we use the notation $X \subset Y$ provided

$$\|u\|_Y \leq k \, \|u\|_X \; for \; every \; u \; \in X \; and \; k > 0.$$

The geometrical interpretation of the Definition 2.1.1 points out a metric relation between the spaces X and Y, i.e., the metric of the space Y is less or equal than the metric of the space X times a positive constant *k* for every element of the embedded space.

Definition 2.1.2. Let X and Y be Banach spaces, and $X \subset Y$. The space X is *compactly embedded* in Y, using the notation $X \subset\subset Y$, if the two following conditions are satisfied

(i) $\|u\|_Y \leq k \, \|u\|_X \; for \; every \; u \; \in X \; and \; k > 0$

and

(ii) each bounded sequence in X is *pre-compact* in Y.

These two definitions are fundamental for our application of the previous material in the study of our problem and we will refer to them quite often in Chapter II.

2.2 Useful inequalities

In this section we introduce a collection of fundamental inequalities which are employed mainly in Chapter II of this work.
The proofs of those are given in Appendix B.2 in [5].

a. Cauchy's inequality.

$$ab \leq \frac{\alpha^2}{2} + \frac{b^2}{2}$$

where $a, b \in \Re$.

b. Cauchy's inequality with ε.

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

where $a, b > 0, \varepsilon > 0$.

c. Young's inequality. Let $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{\alpha^p}{p} + \frac{b^q}{q}$$

where $a, b > 0$.

d. Young's inequality with ε. For $a, b > 0, \varepsilon > 0$, and the previous conditions on p, q we have

$$ab \leq \varepsilon a^p + (\varepsilon p)^{-q/p} q^{-1} b^q.$$

e. Hölder's inequality. Assume $1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^{p}(\Omega)$, $v \in L^{q}(\Omega)$, we have

$$\int_{\Omega} |uv| \, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

f. General Hölder's inequality. Let $1 \leq p_1 \leq p_2 \leq \cdots \leq p_n \leq \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$, and assume $u_k \in L^{p_k}(\Omega)$ for $k = 1, \cdots, n$. Then

$$\int_{\Omega} |u_1 \cdots u_n| \, dx \leq \prod_{k=1}^{n} \|u_k\|_{L^{p_k}(\Omega)}.$$

g. Minkowski's inequality. Assume $1 \leq p \leq \infty$ and $u, v \in L^p(\Omega)$. Then

$$\|u+v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Remark. Inequality g is the *triangle inequality* for the $\| \quad \|_{L^p(\Omega)}$ defined in subsection 1.3, Proposition (a).

h. Interpolation inequality for L^p-norms. Assume $1 \leq s \leq r \leq t \leq \infty$, $\frac{1}{r} = \frac{\theta}{s} + \frac{(1-\theta)}{t}$ and $u \in L^s(\Omega) \cap L^t(\Omega)$. Then $u \in L^r(\Omega)$, and

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{L^s(\Omega)}^{\theta} \|u\|_{L^t(\Omega)}^{1-\theta}.$$

i. Gronwall's inequality (differential form).

(a) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for almost every *t* the differential inequality

$$\frac{d}{dt}\eta(t) \leq \varphi(t)\eta(t) + \psi(t)$$

where $\varphi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \varphi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]$$

for all $t \in [0, T]$.

(b) More precisely, if

$$\eta'(t) \leq \varphi(t)\eta(t) \text{ on } [0, T] \text{ and } \eta(0) = 0$$

then

$$\eta \equiv 0 \text{ on } [0, T].$$

2.3 Sobolev inequalities[3]

According to our discussion in subsection 1.3, the elements of *Sobolev spaces* are *locally summable functions* with *weak derivatives* that belong to the *Lebesgue spaces*. Since the importance of changing the spaces in our problem is rising from the "validity" of our invoked functions, the query about the natural extension of the functions which belong to *Sobolev spaces* in other spaces automatically brings the embeddings theorems back to our mind.

In this section we will therefore give the so-called "*Sobolev inequalities*" in order to define embeddings of various *Sobolev spaces* into others.

3. See more in Section 5.6 in [5].

Theorem 2.3.1 (Gagliardo-Nirenberg-Sobolev inequality). Let $1 \leq p < n$ and the *Sobolev* conjugate of *p* defined as

$$p^* \coloneqq \frac{np}{n-p}, \qquad p^* > p.$$

Then, there exists a constant *K*, depending only on *p* and *n*, such that

$$\|u\|_{L^{p^*}(R^n)} \leq K\|Du\|_{L^p(R^n)}$$

for all $u \in C_c^1(\mathfrak{R}^N)$.

In Theorem 2.3.1 the quantity "*Du*" denotes the *gradient vector of u*, i.e., the *partial derivatives* of *u* with respect to its *spatial arguments*; $Du \coloneqq (u_{x1}, \cdots, u_{xn})$, and the space $C_c^1(\mathfrak{R}^N)$ denotes the space of functions which, together with their *first-order derivatives* are *continuous* in $\mathfrak{R}^N$ and has *compact support*[4] in it.

Theorem 2.3.2 (Estimates for $W^{1,p}, 1 \leq p < n$). Assume Ω is a bounded, open subset of $\mathfrak{R}^N$, and suppose $\vartheta\Omega$ (the boundary of Ω) is C^1. Assume $1 \leq p < n$, and $u \in W^{1,p}(\Omega)$. Then $u \in L^{p^*}(\Omega)$, with the estimate

$$\|u\|_{L^{p^*}(\Omega)} \leq K\|u\|_{W^{1,p}(\Omega)}$$

the constant *K* depending only on *p*, *n*, and Ω.

Definition 2.3.3. We denote by $W_0^{k,p}(\Omega)$ the *closure* of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.
Thus $u \in W_0^{k,p}(\Omega)$ *if and only if* there exist functions $u_m \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$.
The space $W_0^{k,p}(\Omega)$ consists of those function that belong to $W^{k,p}(\Omega)$ and have the following property

$$\text{“}D^a u = 0 \text{ on } \vartheta\Omega\text{” for all } |a| \leq k-1.$$

Theorem 2.3.4 (Estimates for $W_0^{1,p}, 1 \leq p < n$). Assume Ω is a bounded, open subset of $\mathfrak{R}^N$. Suppose also $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p < n$. Then we have the following estimate

$$\|u\|_{L^q(\Omega)} \leq K\|Du\|_{L^p(\Omega)}$$

for each $q \in [1, p^*]$, with the constant *K* depending only on *p*, *n*, *q* and Ω.

4. Given a function $u \in \Omega$, its support is defined to be

$$supp(u) \coloneqq \overline{\{\boldsymbol{x} \in \Omega |\, u(\boldsymbol{x}) \neq 0\}}.$$

We say a function *u* has *compact support* if *supp(u)* is a proper subset of Ω, i.e., if *supp(u)* is bounded; that means that there exist $a, b \in \Omega$ such that $supp(u) \subseteq [a, b]$.

Remark. For all $1 \leq p \leq \infty$,

$$\|u\|_{L^p(\Omega)} \leq K\|Du\|_{L^p(\Omega)}.$$

This estimate is called *Poincaré's inequality*.
In view of this estimate, on $W_0^{1,p}(\Omega)$ the norm $\|Du\|_{L^p(\Omega)}$ is equivalent to $\|u\|_{W^{1,p}(\Omega)}$, if Ω is bounded.

Assume $\Omega \subset \mathfrak{R}^N$ is open and $0 < \gamma \leq 1$. We say a function *u* satisfying the following inequality is said to be *Hölder continuous with exponent γ.*

$$|u(\boldsymbol{x}) - u(\boldsymbol{y})| \leq k\|\boldsymbol{x} - \boldsymbol{y}\|^\gamma, \text{ for } \boldsymbol{x}, \boldsymbol{y} \in \Omega.$$

Definition 2.3.5. For $m \in \mathbb{Z}_+$ and $\gamma \in (0,1]$, we define the *Hölder space*

$$C^{m,\gamma}(\bar{\Omega}) \coloneqq \{u \in C^m(\bar{\Omega})|\ \vartheta^\alpha u \in C^{0,\gamma}(\bar{\Omega})\ \forall \alpha \ with\ |a| = m\ \}.$$

This is a *Banach space* with respect to the norm

$$\|u\|_{C^{m,\gamma}(\bar{\Omega})} \coloneqq max_{|\alpha|\leq m}\|\vartheta^\alpha u\|_{C(\bar{\Omega})}$$

$$+ \sum_{|a|=m} sup\left\{\frac{|\vartheta^\alpha u(\boldsymbol{x}) - \vartheta^\alpha u(\boldsymbol{y})|}{\|\boldsymbol{x} - \boldsymbol{y}\|^\gamma}\,|\ \boldsymbol{x}, \boldsymbol{y}\ \in\ \Omega, \boldsymbol{x} \neq \boldsymbol{y}\right\}$$

where $\|u\|_{C(\bar{\Omega})} \coloneqq \sup_{x\in\Omega}|\mathrm{u(x)}|$.

So, the *space* $C^{m,\gamma}(\bar{\Omega})$ consists of those functions *u* that are *m-times continuously differentiable* and whose *k^{th}-partial derivatives* are *bounded* and *Hölder continuous with exponent γ.*

Theorem 2.3.6. Assume $n < p \leq \infty$. Then there exists a *constant K*, depending only on *p* and *n*, such that

$$\|u\|_{C^{0,\gamma}(R^N)} \leq K\|u\|_{W^{1,p}(R^N)}$$

for all $u \in C^1(\mathfrak{R}^N)$, where $\gamma \coloneqq 1 - n/p$.

Theorem 2.3.7 (General *Sobolev* inequalities). Assume $\Omega \subset \mathfrak{R}^N$ is *bounded* and *open*, and suppose $\vartheta\Omega$ (the boundary of Ω) is C^1. Assume also $u \in W^{k,p}(\Omega)$.

(i) *If*

$$k < n/p$$

then $u \in L^q(\Omega)$, *where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

and we have also the estimate

$$\|u\|_{L^q(\Omega)} \leq K\|u\|_{W^{k,p}(\Omega)},$$

where the constant K depending only on k, p, n and Ω.

(ii) *If*

$$k > n/p$$

then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\bar{\Omega})$, *where*

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, if \frac{n}{p} is\ not\ an\ integer \\ any\ positive\ number < 1, if \frac{n}{p} is\ an\ integer. \end{cases}$$

In addition, we have the following estimate

$$\|u\|_{C^{k-\frac{n}{p}-1,\gamma}(\bar{\Omega})} \leq K\|u\|_{W^{k,p}(\Omega)},$$

where the constant K depending only on k, p, n, γ and Ω.

3. Calculus on normed vector spaces[5]

Calculus has always been the pioneer of change in our modern theories of science as with its notion of differentiation and integration we gave shape in uncounted physical concepts. In the present section we will extend the properties of calculus beyond the two-dimensional space and propose the equivalent theorems in the whole space of $\mathfrak{R}^N$.

3.1 Smoothness of the boundary

Sobolev spaces among the many properties they have, require also specific smoothness-arguments concerning the boundary of the domain for their properties to hold.

In order to explain the things a little further let us assume a vector field in two-dimensional space. We will make the assumption that the "source" of the field takes place in the origin of our space and that we have an isotropic propagation, i.e., equally distribution of propagation among the directions. Obviously, the field has the shape of a circle with the vectors pointing outward propagating the action (or *flow*) equally throughout the space, like a wave.

When referring to the regularity of the boundary of a domain we mean the continuity of propagation in the limits (of the domain) as defined by the "transmission-rules" of the setting framework of action, i.e., the continuity of the derivatives of points belonging to the boundary.

Definition 3.1.1. Assume $\Omega \subset \mathfrak{R}^N$ be open and bounded, $k \in \{1, 2, \cdots\}$. We say the boundary $\vartheta\Omega$ is C^k if for each point $\boldsymbol{x}^0$ belonging to the boundary there exists $r > 0$ and a C^k-function $g : \mathfrak{R}^{N-1} \to \mathfrak{R}$ such that upon a transformation of the coordinate system, if necessary, we have

$$\Omega \cap B(\boldsymbol{x}^0, r) = \{x \in B(\boldsymbol{x}^0, r) |\, x_N > g(x_1, \cdots, x_{N-1})\}.$$

Here $B(\boldsymbol{x}^0, r)$ denotes the *N-dimensional ball* centered at $\boldsymbol{x}^0$ with radius r.

5. See more on Chapter 7 of [4], Appendix C in [5], and for a general reading in [11].

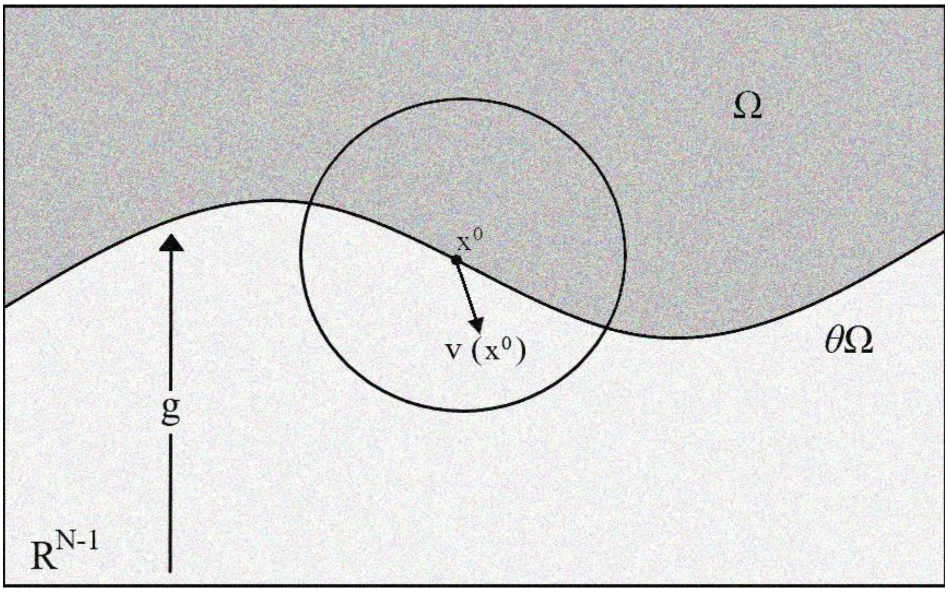

Figure 1. Smoothness of the boundary of $\Omega \subset \mathfrak{R}^N$.

Remark. (i) If $\vartheta\Omega$ is C^1, then along the boundary of Ω we can define the *outward pointing unit normal vector field* $\boldsymbol{v} = (v^1, \cdots, v^N)$.
The *unit normal* at any point $\boldsymbol{x}^0$ belonging to the boundary is $\boldsymbol{v}(\boldsymbol{x}^0) = (v_1^0, \cdots, v_N^0)$.
(ii) Assume *u* belonging to $C^1(\bar{\Omega})$. We define by

$$\vartheta u / \vartheta v \coloneqq v \cdot Du$$

the (*outward*) *normal derivative of u*.

3.2. Gauss-Green Theorem

By our previous notes on *smoothness* and *outward normal vectors* the integration of a function belonging to C^1 over the boundary of the domain is properly defined.
In this subsection we will take Ω to be a bounded, open subset of $\mathfrak{R}^N$ and the boundary of Ω to be C^1.

Theorem 3.2.1 (Gauss-Green Theorem). (i) Assume *u* belonging to $C^1(\bar{\Omega})$. Then

$$\int_{\Omega} u_{x_i}\, dx = \int_{\vartheta\Omega} u\, v^i dS \ (i = 1, \cdots, N).$$

where u_{x_i} denotes the partial derivatives of *u* with respect to the spatial dimensions, and v^i the *outward pointing unit normal vector field*, integrated with respect to the boundary of Ω.

(ii) We also have

$$\int\limits_{\Omega} div\, \boldsymbol{u}\, dx = \int_{\vartheta\Omega} \boldsymbol{u} \cdot v\, dS$$

for each *vector field* $\boldsymbol{u} = (u^1, \cdots, u^N) \in C^1(\bar{\Omega}\,; \mathfrak{R}^N)$, which is referred as the *Divergence Theorem*. ■

Theorem 3.2.2 (Integration by parts). Assume $u, v \in C^1(\bar{\Omega})$. Therefore

$$\int\limits_{\Omega} u_{x_i} v\, dx = -\int\limits_{\Omega} v_{x_i} u\, dx + \int_{\vartheta\Omega} uvv^i\, dS\ (i = 1, \cdots, N).$$

Proof. We begin by integrating over Ω the quantity $(uv)_{x_i}$.

$$\int\limits_{\Omega} (uv)_{x_i}\, dx \stackrel{\text{def}}{=} \int\limits_{\Omega} \left(u_{x_i} v + v_{x_i} u\right) dx = \int\limits_{\Omega} u_{x_i} v\, dx + \int\limits_{\Omega} v_{x_i} u\, dx \quad (1)$$

where the first equality is valid from the *product rule*.
Applying the previous theorem to uv, we get

$$\int\limits_{\Omega} (uv)_{x_i}\, dx = \int_{\vartheta\Omega} uvv^i\, dS \quad (2)$$

From (1) and (2) we take *the integration by parts formula*. Q.E.D.

Theorem 3.2.3 (Green's formulas). Assume $u, v \in C^2(\bar{\Omega})$. Then

$(i)\ \int_{\Omega} \Delta u\, dx = \int_{\vartheta\Omega} \frac{\vartheta u}{\vartheta v}\, dS,$

$(ii)\ \int_{\Omega} Du \cdot Dv\, dx = -\int_{\Omega} u\Delta v\, dx + \int_{\vartheta\Omega} \frac{\vartheta v}{\vartheta v} u\, dS,$

$(iii)\ \int_{\Omega} u\Delta v - v\Delta u\, dx = \int_{\vartheta\Omega} \frac{\vartheta v}{\vartheta v} u - \frac{\vartheta u}{\vartheta v} v dS,$

where D denotes the *Hamilton operator*

$$\nabla \stackrel{\text{def}}{=} \frac{\vartheta}{\vartheta x_i}\ (i = 1, \cdots, N). \blacksquare$$

4. Measure Theory

In subsection 1.3 we referred to *Lebesgue integrable functions* and defined the Lebesgue spaces (see *Def. 1.3.1*). In this section we make a brief outline of some fundamental properties of measure theory and more specifically of the concepts of *Lebesgue measure* and *Lebesgue integration*. For a general studying on measure theory see [2] or Lecture 6 -Multiple Integrals: Lebesgue Integration- in [12]. For a short introduction see Chapter 1 in [4] or Appendix E in [5].

4.1 Lebesgue Measure

According to the dimensionality of the space the *measure* appears as a generalization of the "length", "area", "volume" or an affinal generalization.

Definition 4.1.1. Assume Ω is an open set and Φ a closed set. We define the *Lebesgue measure* of the sets Ω and Φ respectively by

$$m\Omega = \int_{\Omega} dv \ \ and\ m\Phi = \int_{\Phi} dv.$$

Theorem 4.1.2. Assume the set E lying in an open set Ω. We say E is *measurable* if there exist a sequence of closed sets F_k included in E and a sequence of open sets Ω_k containing E, such that

$$m(\Omega_k - F_k) \to 0 \text{ as } \ k \to \infty. \ \blacksquare$$

For the proof of Theorem 4.1.2 see pp.104-105 in [12].

Remark. Assume *F* are closed sets included in *E* and *Ω* are open sets containing *E*. If the set *E* is measurable, then

$$mE = \sup_{F \subseteq E} mF = \inf_{E \subseteq \Omega} m\Omega.$$

Proposition 4.1.3. For any two measurable sets E_1 and E_2, the following relation holds

$$mE_1 + mE_2 = m(E_1 + E_2) + m(E_1 E_2),$$

where by $E_1 E_2$ is denoted their intersection. ■

Hence, we see that the sum and intersection of two measurable sets are always measurable.

Remark. If there are open sets Ω_k containing E with $\inf m\Omega_k = 0$, then according to the previous remark, $mE = 0$. *Conversely, any set having zero measure can be included in an open set with measure as small as we please.*

Theorem 4.1.4. If on a bounded open set Ω a sequence of measurable sets $\{E_k\}_k$ is given with no common points, then the sum of these sets $E = \sum_k E_k$ is measurable with

$$mE = \sum_{k=1}^{\infty} mE_k. \blacksquare$$

For the proof see pp.112 in [12].

4.2 Lebesgue Integration

Following the concept of measurable sets, we procced with the definition of the measurable functions.

Definition 4.2.1. Assume Ω is a bounded open set and *f* a function defined on that set. We say *f* is measurable *if and only if* there exist closed sets F_k with measure close to that of Ω according to our will, on which *f* is continuous, and

$$m\Omega - mF_k \leq \varepsilon,$$

where ε is any positive number.

Definition 4.2.2. If a non-negative function *f* has an *inner integral*[6] in an open set Ω, then we say *f* is *integrable,* or *summable* in the Lebesgue sense on the domain Ω, with its inner integral representing then its *Lebesgue integral* with the notation of an ordinary integral.

6. Assume *f* an arbitrary non-negative function defined on a bounded open set Ω. Consider all the closed sets F included in Ω on which *f* is continuous and define the upper bound of the integrals of *f* taken over the sets F; $sup_F \int_F f\, d_U$.
We shall call this upper bound, if it exists and is finite, the inner integral on the set Ω of *f* and we shall use the notation

$$(in)\int_{\Omega} f\, dv.$$

We will complete this outline by giving two of the most important theorems that characterize Lebesgue integration.

Theorem 4.2.3 (Lebesgue Dominated Convergence Theorem). Suppose $\{f_k\}_k$ is *a sequence of summable function* converging almost everywhere in the open set Ω to a limit function *f*. If the *functions* $|f_k| \leq g$, where g *a certain summable function*, then the function *f* is summable with

$$\int_{\Omega} f\, dv = \lim_{k\to\infty} \int_{\Omega} f_k dv\,.$$

Theorem 4.2.4 (Fubini's Theorem). Assume $\Omega_1 \subset \Re^{N_1}$ and $\Omega_2 \subset \Re^{N_2}$ are *Lebesgue measurable sets*, and let *f* be a *summable function* defined on $\Omega = \Omega_1 \times \Omega_2$. Then for almost every $\boldsymbol{x} \in \Omega_1$, the function $f(\boldsymbol{x},\cdot)$ is *Lebesgue integrable* on Ω_2, $\int_{\Omega_2} f(\boldsymbol{x},\boldsymbol{y})\, dy$ is integrable on Ω_1, and

$$\int_{\Omega_1} \left[\int_{\Omega_2} f(\boldsymbol{x},\boldsymbol{y})\, dy\right] dx = \int_{\Omega} f(\boldsymbol{x},\boldsymbol{y})\, dxdy.$$

Similarly, for almost every $\boldsymbol{y} \in \Omega_2$, the function $f(\cdot,\boldsymbol{y})$ is *Lebesgue integrable* on Ω_1, $\int_{\Omega_1} f(\boldsymbol{x},\boldsymbol{y})\, dx$ is integrable on Ω_2, and

$$\int_{\Omega_2} \left[\int_{\Omega_1} f(\boldsymbol{x},\boldsymbol{y})\, dx\right] dy = \int_{\Omega} f(\boldsymbol{x},\boldsymbol{y})\, dxdy.$$

From the validity of the proof of the above two theorems follows the admissibility of passing to the limit under the integral sign, the criterion for the convergence in the mean of a sequence of function, and the possibility of changing the order of integration in a multiple integral.

5. Semigroup Theory[7]

In our previous concerns we briefly talked about the recasting of an PDE in an operator equation, with the operator encoding the structure of our original equation. The validity of our problem upon the recasting procedure must be preserved, i.e., the information regarding boundary or initial conditions, in addition to the prescription of the arguments of our equation must be properly redefined without being affected. *Sine qua non* of this treatment is the construction of a *time-dependent family of operators* in order to represent the evolution of our system in proper manner with respect to our initial conditions.

In this section we construct the pillars of this treatment and outline some of the most vital elements of the theory of semigroups.

5.1 Definitions and properties

We assume X is a real Banach space, and we consider the *initial-value* problem

$$\begin{cases} \dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t), \;\; t \geq 0 \\ \boldsymbol{u}(0) = u, \end{cases}$$

where $\cdot = \frac{d}{dt}$, $u \in X$ is our initial data, and A is a linear operator. Suppose, also, that the *domain* of A, $\mathcal{D}(A)$, is a linear subspace of X. Therefore, we have

$$A : \mathcal{D}(A) \subset \mathrm{X} \to \mathrm{X}.$$

Our intention is to study the *existence* and *uniqueness* of a solution of the following form

$$\boldsymbol{u} : [0, \infty) \to \mathrm{X}.$$

We propose for the moment that $\boldsymbol{u} : [0, \infty) \to \mathrm{X}$ is indeed a solution of the initial-value problem and that for each initial data u there exists a unique solution.

7. See Chapter 2 and 6 in [4], or Chapter 7 in [5].

Definition 5.1.1. We denote the solution as

$$\boldsymbol{u}(t) \coloneqq S(t)u,\ \ t \geq 0,$$

to represent the dependence of our solution on the initial data.
By this notation we may regard $S(t)$ as a *time-dependent mapping from X into itself.*

Proposition. (a) The family of operators $\{S(t)\}_{t\geq 0}$ is linear with

(1) $$S(0)u = u \text{ for each } u \in X,$$

i.e., $S(0) = I$, where $I: X \to X$ is the *identity mapping*.
(b) Our initial-value problem has a unique solution for each initial data, i.e.,

(2) $$S(t+s)u = S(t)S(s)u = S(s)S(t)u \ \ (t, s \geq 0, u \in X).$$

The solution of our problem as we denoted is $\boldsymbol{u}(t) \coloneqq S(t)u,\ \ t \geq 0$.
We have $\boldsymbol{u}(t_0) = S(t_0)u$ *and* $S(t)u(t_0)$ *is the solution of the differential equation on* $[t_0, T]$ *with the initial condition* $u(t_0)$ *at* t_0*. By the uniqueness of the solution,* $S(t)u(t_0) = \boldsymbol{u}(t+t_0)$*, i.e.,* $S(t_1)S(t_0)u = S(t_1+t_0)u$*, since* $u \in X$ *is arbitrary,* $S(t_1+t_0) = S(t_1)S(t_0)$. Q.E.D.
(c) For each $u \in X$ the mapping $t \to S(t)u$ is continuous from $[0, \infty)$ into X.

Definition 5.1.2. *We say a family of bounded linear operators* $\{S(t)\}_{t\geq 0}$ *mapping X into X is a semigroup if and only if the conditions of the above proposition are satisfied.*

Remark. In addition, we say $\{S(t)\}_{t\geq 0}$ is a *contraction semigroup if and only if*

$$\|S(t)\| \leq 1 \ \ (t \geq 0),$$

where $\|\cdot\|$ denoting the *operator norm* defined in section 1 (see Def. 1.4.4). Thus

$$\|S(t)u\| \leq \|u\|,\ \ t \geq 0, u \in X.$$

Definition 5.1.3. We say $A : \mathcal{D}(A) \subset X \to X$ is the *infinitesimal generator* of the semigroup $\{S(t)\}_{t\geq 0}$ with

$$\mathcal{D}(A) \coloneqq \left\{u \in X \mid \lim_{t\to 0^+} \frac{S(t)u-u}{t} \ \textit{exists in } X\right\}$$

and

$$Au \coloneqq \lim_{t\to 0^+} \frac{S(t)u-u}{t} \ \ (u \in \mathcal{D}(A)).$$

Theorem 5.1.4. Assume $u \in \mathcal{D}(A)$. Therefore, we have

(i) $S(t)u \in \mathcal{D}(A)$ for each $t \geq 0$.

(ii) $AS(t)u = S(t)Au$ for each $t \geq 0$.

(iii) The mapping $t \rightarrow S(t)u$ is differentiable for each $t > 0$.

(iv) $\frac{d}{dt}S(t)u = AS(t)u,\ t > 0$. ■

For the proof see pp. 435-436 in [5].

Remark. From property (iv) and the fact that $t \rightarrow AS(t)u = S(t)Au$ is continuous, the mapping $t \rightarrow S(t)u$ is C^1 in $(0, \infty)$, if $u \in \mathcal{D}(A)$.

Theorem 5.1.5. Under the previous assumption we have

(i) The *domain of the operator A* is *dense* in X

and

(ii) the operator *A* is *closed*, i.e., if for any *sequence* $\{v_k\} \subset \mathcal{D}(A)$, $v_k \rightarrow v$ and $A(v_k) \rightarrow w$ as $k \rightarrow \infty$, we have $v \in \mathcal{D}(A)$ and $w = A(v)$. ■

For the proof see pp. 436-437 in [5].

Definition 5.1.6. (a) We say a *real number* λ belongs to the *resolvent set of A*, *ρ(A)*, provided the operator

$$\lambda I - A : \mathcal{D}(A) \rightarrow X$$

is *bijective*.

(b) If $\lambda \in \rho(A)$, the *resolvent operator* R_λ : X → X is defined by

$$R_\lambda u \coloneqq (\lambda I - A)^{-1}u.$$

Remark. According to the Closed Graph Theorem[8], R_λ : X → $\mathcal{D}(A) \subseteq$ X is a bounded linear operator, and furthermore for $u \in \mathcal{D}(A)$ we have, $AR_\lambda u = R_\lambda Au$.

The importance of defining the *resolvent set* and the *resolvent operator* is rising from the existence of solutions of equations of the following form

$$(\lambda I - A)u = f,$$

where f is a given function defined on X and λ a real positive number.

Obviously, the solution takes the form, $u = (\lambda I - A)^{-1}f$ and therefore the consideration of the set of all real numbers λ for which such a solution is valid, i.e., the inverse operator $(\lambda I - A)^{-1}$ exists on X to $\mathcal{D}(A) \subseteq$ X, defines the *resolvent set.*

Definition 5.1.7. The set that remains after the exclusion of the *resolvent set* from the set of real numbers is called the *spectrum* of A, i.e., $\sigma(A) = \Re - \rho(A)$.

8. **(Closed Graph Theorem)**: Assume $A : X \rightarrow Y$ be a closed, linear operator. Then A is bounded.

Theorem 5.1.8. (a) If $\lambda, \mu \in \rho(A)$, we have

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \quad (\textit{resolvent identity})$$

and

$$R_\lambda R_\mu = R_\mu R_\lambda.$$

(b) If $\lambda > 0$, then $\lambda \in \rho(A)$,

$$R_\lambda u = \int_0^\infty e^{-\lambda t} S(t) u \, dt \quad (u \in X),$$

and so $\|R_\lambda\| \leq \frac{1}{\lambda}$. ■

Thus, *the resolvent operator is the Laplace transform*[9] *of the semigroup* (see Example 8, pp. 203 in [5]). For the proof see pp. 438-439 in [5].

So far, we talked about semigroups, and the relation between them and the operators of the posed problem. A significant account about relating these two concepts is which operator generates a contraction semigroup. In the following theorem which is also referred as *Hille-Yosida Theorem* we give the necessary conditions.

Theorem 5.1.9. Assume A to be a closed, densely-defined linear operator on X. Then we say, *A generates a contraction semigroup* $\{S(t)\}_{t\geq 0}$ *if and only if*

$$(0, \infty) \subset \rho(A) \text{ and } \|R_\lambda\| \leq \frac{1}{\lambda} \text{ for } \lambda > 0. \blacksquare$$

For the proof see pp. 439-441 in [5].

8. Assume $u \in L^1(\mathfrak{R}^+)$. We define its *Laplace transform* using the notation $\mathscr{L}u = u$ as

$$u(s) := \int_0^\infty e^{-st} u(t) dt \quad (s \geq 0).$$

6. The Banach fixed-point theorem

In this section we investigate the existence of solutions to *operator equations* lacking the property of *linearity* and which take the following form

$$\mathfrak{T}(u) = u,\ u \in \mathrm{V},$$

where the space V is a subset of a *real Banach space* X, and the operator $\mathfrak{T}$ is a mapping from V to X.

Obviously, according to the equation, the points that satisfy the above condition remain *unchanged* under the action of $\mathfrak{T}$, i.e., the solutions of this equation may be regarded as *fixed-points* of the operator $\mathfrak{T}$.

It should be also mentioned that if we take a closer look on our problem then we will notice that the analysis of the solvability is based on the analysis of the operator, as the fixed-points of an operator *may not be* fixed for another operator, i.e., the treatment should be made on the operator itself.

By this statement we begin our analysis assuming X to be a real Banach space equipped with the norm $\|\cdot\|_X$, V to be a subset of X, and $\mathfrak{T}$ an operator from V to X.

Definition. We say an operator $\mathfrak{T} : \mathrm{V} \subset \mathrm{X} \to \mathrm{X}$ is *contractive* with *contractivity-constant* $0 \leq \alpha < 1$ *if*

$$\|\mathfrak{T}(u) - \mathfrak{T}(v)\|_{\mathrm{X}} \leq \alpha\, \|u - v\|_{\mathrm{X}} \quad \forall u, v \in \mathrm{V}.$$

Remark. From a geometrical point of view, the above inequality points out that the distance of the image of two elements is less or equal than the distance of the elements itself, i.e., a "*distance-contraction*" between two elements is accomplished.

Theorem. (a) *Existence and uniqueness:*

Assume V is a non-empty closed set in a real Banach space X, and also, that $\mathfrak{T} : \mathrm{V} \to \mathrm{V}$ is a *contractive* mapping with *contractivity-constant* α, $0 \leq \alpha < 1$. Therefore, there exists a unique $u \in \mathrm{V}$ such that $\mathfrak{T}(u) = u$.

If we are also interested in *approximating* the solution of the *fixed-point problem* by following an *iterative method* then we get also the following result;

(b) *Convergence and error estimates of the iteration:*

For any initial guess $u_0 \in$ V, the sequence $\{u_k\} \subset$ V defined by the *iteration formula* $u_{k+1} = \mathfrak{T}(u_k)$, $k = 0, 1, \ldots$, converges to u; $\|u_k - u\|_X \to 0$ as $k \to \infty$. For the error estimates, we have the following valid bounds:

(i) $\|u_k - u\|_X \leq \frac{\alpha^k}{1-\alpha} \|u_0 - u_1\|_X$,

(ii) $\|u_k - u\|_X \leq \frac{\alpha}{1-\alpha} \|u_{k-1} - u_k\|_X$,

(iii) $\|u_k - u\|_X \leq \alpha \|u_{k-1} - u\|_X$.

Proof. (a) Since $\mathfrak{T}$ is a mapping from V to V, the sequence $\{u_k\}$ is well-defined. Firstly, we prove that $\{u_k\}$ is a *Cauchy-sequence*. From the *contractivity* of the operator $\mathfrak{T}$ we have

$$\begin{aligned} d(u_{k+1}, u_k) &= d(\mathfrak{T}(u_k), \mathfrak{T}(u_{k-1})) \\ &= \|\mathfrak{T}(u_k) - \mathfrak{T}(u_{k-1})\|_X \\ &\leq \alpha \, \|u_k - u_{k-1}\|_X \\ &= a d(\mathfrak{T}(u_{k-1}), \mathfrak{T}(u_{k-2})) \\ &\leq \alpha^2 \, \|u_{k-1} - u_{k-2}\|_X \\ &\;\;\vdots \\ &\leq \alpha^k \, d(u_1, u_0). \end{aligned}$$

Therefore, for any $l \geq k \geq 1$,

$$d(u_l, u_k) \leq \sum_{j=0}^{l-k-1} d\big(u_{k+j+1}, u_{k+j}\big)$$

$$\leq \sum_{j=0}^{l-k-1} \alpha^{k+j} \, d(u_1, u_0)$$

$$\leq \frac{a^k}{1-a} \, d(u_1, u_0).$$

.

Since $0 \leq \alpha < 1$, $d(u_l, u_k) \to 0$ as $l, k \to \infty$; thus $\{u_k\}$ is a *Cauchy-sequence*. From the fact that V is a closed set in the Banach space X, $\{u_k\}$ converges to an element $u \in$ V. Taking the limit $k \to \infty$ in $u_{k+1} = \mathfrak{T}(u_k)$, we see that $u = \mathfrak{T}(u)$ by the continuity of $\mathfrak{T}$, i.e., we proved that u is a fixed-point of $\mathfrak{T}$.

To complete the first part of our proof we need to show the *uniqueness of our solution*. For this, suppose that $u_1, u_2 \in$ V are both fixed-points of the operator $\mathfrak{T}$. Then from this statement we obtain; $u_1 = \mathfrak{T}(u_1)$ and $u_2 = \mathfrak{T}(u_2)$ or

$$u_1 - u_2 = \mathfrak{T}(u_1) - \mathfrak{T}(u_2).$$

Taking the norm on both quantities we get

$$\|u_1 - u_2\|_X = \|\mathfrak{T}(u_1) - \mathfrak{T}(u_2)\|_X \leq \alpha \, \|u_1 - u_2\|_X$$

which implies since $0 \leq \alpha < 1$, that $d(u_1, u_2) = 0$, i.e., $u_1 = u_2$. So, we proved that a fixed-point of a contractive mapping is unique.

(b) In the second part of our proof, we need to prove the validity of the error estimates. By our previous calculations we showed that

$$d(u_l, u_k) \leq \frac{a^k}{1-a} \, d(u_1, u_0).$$

Letting $l \to \infty$ and using the convergence $u_l \to u$ and the (iii)-property of the distance between two elements we get the first error estimate.
From

$$\|u_k - u\|_X = \|\mathfrak{T}(u_{k-1}) - \mathfrak{T}(u)\|_X \leq \alpha \, \|u_{k-1} - u\|_X$$

we obtain the third estimate. Using this, together with

$$\|u_{k-1} - u\|_X \leq \|u_{k-1} - u_k\|_X + \|u_k - u\|_X$$

we get the second estimate. Q.E.D.

This theorem is known as the *Banach fixed-point theorem* or as *contractive mapping theorem*.

7. The Galerkin method

In many problems involving nonlinear partial differential equations, the use of numerical methods is the *lapis primus* of our study. In this section we will make a briefly introduction to the *Galerkin method* and discuss *the approximation techniques* used for the *weak-solutions* of a problem. For a general study on the Galerkin method and some of its variants see Chapter 9 in [4]. For an application on hyperbolic initial/boundary-value problems see Chapter 7 in [5].

7.1 Introduction

The formulation of a problem with initial, boundary, or a combination of these conditions, concerning the behavior of our unknown function, in the type of a system composed by a partial differential equation and the mentioned conditions is regarded as a *classical formulation*. On the other hand, if we properly remove the smoothness requirements of the solution we derive the *weak formulation* of our problem, which is equivalent to our original one, but more treatable.

To understand the "transition" from *classical* to *weak formulation* let us see two examples of some *linear elliptic boundary-value problems*.

Example 7.1.1. Suppose that we are given the following elliptic boundary-value problem which corresponds to the *Poisson equation*[9] with the *homogeneous Dirichlet boundary condition*

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \quad u = 0 \text{ in } \theta\Omega. \end{cases}$$

To derive the weak formulation, we multiply the differential equation by a *smooth test function*, i.e., by an arbitrary function υ belonging to $C_0^\infty(\Omega)$, and integrate with respect to Ω.

9. The *Poisson equation* in the nonlinear form $-\Delta u = f$ is the *non-homogeneous Laplace equation* which can be used to describe among others *steady state heat conduction*, *electrostatics*, *gravity potential in free space*, etc. See more in [5].

This leads to

$$-\int_{\Omega} \Delta uv\, dx = \int_{\Omega} fv\, dx\,.$$

According to the *Green's second formula* we have

$$\int_{\Omega} \nabla u \nabla v\, dx - \int_{\vartheta\Omega} \frac{\vartheta v}{\vartheta v} u\, dS = \int_{\Omega} fv\, dx,$$

and by *the compact support of* v ($v = 0, in\ \theta\Omega$) the equation is reduced to

$$\int_{\Omega} \nabla u \nabla v\, dx = \int_{\Omega} fv\, dx\,.$$

The term on the left-hand side of the equation points out that the functions u, v should belong to the *Sobolev space* $H_0^1(\Omega)$ *- instead of the space* $C^2(\Omega) \cap C(\bar{\Omega})$ *which is the proper space in our classical formulation* - and from the right-hand side term $f \in L^2(\Omega)$ in order for the last equation to have a meaning. Under these assumptions we have a *weak formulation* of our problem:

$$u, v \in H_0^1(\Omega), \int_{\Omega} \nabla u \nabla v\, dx = \int_{\Omega} fv\, dx,\ \ f \in L^2(\Omega).$$

If in addition we assume $\mathrm{X} = \mathrm{H}_0^1(\Omega)$, $\alpha(\ ,\ ) : X \times X \to \Re$, the *bilinear form* defined by

$$\alpha(u, v) = \int_{\Omega} \nabla u \nabla v\, dx \quad for\ u, v \in X,$$

and $l : X \to \Re$, the *linear functional* defined by

$$l(v) = \int_{\Omega} fv\, dx\ for\ v \in X,$$

the *weak formulation* of the problem takes the form

$$\alpha(u, v) = \ l(v)\ \ \forall v \in X,$$

where $u \in X$ is the solution.

Example 7.1.2. Consider the *Helmholtz equation* with the *homogeneous Dirichlet boundary condition*

$$\begin{cases} -\Delta u + u = f \text{ in } \Omega, \\ \qquad u = 0 \text{ in } \vartheta\Omega. \end{cases}$$

(For a study in *Helmholtz equation* see [5]).

Following the previous procedure, we derive the equation

$$-\int_{\Omega}(\Delta uv + uv)\,dx = \int_{\Omega} fv\,dx.$$

As before, using *Green's second formula* and the *compact support of* v we take

$$\int_{\Omega}(\nabla u \nabla v + uv)\,dx = \int_{\Omega} fv\,dx.$$

Obviously, the validity of this equation holds if $u, v \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$.
Therefore, the *weak formulation* of our problem is

$$u, v \in H_0^1(\Omega), \int_{\Omega}(\nabla u \nabla v + uv)\,dx = \int_{\Omega} fv\,dx, \;\; f \in L^2(\Omega),$$

and if we take into consideration our previous notation we have

$$\alpha(u,v) = l(v),$$

for $X = H_0^1(\Omega)$, $u, v \in X$ and $\alpha(u,v) = \int_{\Omega}(\nabla u \nabla v + uv)\,dx, \;\; l(v) = \int_{\Omega} fv\,dx.$

The significance of the *weak formulation* is that we can use the *Galerkin method*, which we develop in the next subsection, *to approximate the weak solutions of our problem*.

7.2 The method

Assume $X = H_0^1(\Omega)$, $\alpha(\;,\;) : X \times X \to \mathfrak{R}$ a *bilinear form*, and $l : X \to \mathfrak{R}$ a *linear functional*. We consider our previous notation on the *weak formulation* of our problem

$$\alpha(u,v) = l(v) \;\; \forall u, v \in X,$$

and the following two conditions

(i) $\alpha(\;,\;)$ is bounded, i.e., $|\alpha(u,v)| \leq K\|u\|_X\|v\|_X \; \forall u, v \in X,$
and
(ii) V-elliptic, i.e., $\alpha(v,v) \geq k_0\|v\|_X^2 \; \forall v \in X.$

Therefore, according to the *Lax-Milgram Lemma*, our problem has a unique solution.

Even though the existence and uniqueness of the weak solution has been secured by the above lemma, the exact solution may not be easily found due to the infinite-dimensionality of our space. To overcome this “peculiarity” we project our problem to a finite-dimensional subspace and using the *Lax-Milgram Lemma* we secure again the existence and uniqueness of the solution. Thus, assuming $X_N \subset X$ to be our N-dimensional projection we get

$$\alpha(u_N, v) = l(v) \;\; \forall u_N, v \in X_N.$$

We may actually make our work much easier if we recast our problem in the form of a *linear system*. To do this, we take $\{w_i\}_{i=1}^{N}$ to be a basis[10] of X_N, and write

$$u_N = \sum_{j=1}^{N} d_j w_j.$$

Taking $v \in X_N$ to be each of the basis-functions w_i, and substituting to the problem we take

$$a\left(\sum_{j=1}^{N} d_j w_j, w_i\right) = l(w_i)$$

or

$$\left(a\left(w_j, w_i\right)\right)(d_j) = l(w_i)$$

where using the following notation

$A = \left(a\left(w_j, w_i\right)\right) \in \mathfrak{R}^{\mathrm{NxN}} \equiv$ *the stiffness matrix*
$\boldsymbol{d} = (d_j) \in \mathfrak{R}^{\mathrm{N}} \equiv$ *the unknown vector*
$\boldsymbol{b} = l(w_i) \in \mathfrak{R}^{\mathrm{N}} \equiv$ *the load vector*

we derive the *equivalent linear system,* $A\boldsymbol{d} = \boldsymbol{b}$.

Following the prescribed procedure for an expanding sequence of subspaces we can increase the accuracy of approximation, and use *Céa’s lemma*[11] for error estimates.

By this we complete our introduction to the *Galerkin method*, a variant of which we shall see in Chapter II.

10. A basis is a set of independent vectors such that any vector in the space can be written as a linear combination of them. See more on Chapter 1 in [4].
11. See Proposition 9.1.3 in [4].

B. Introduction

Our aim in this work is to study the existence of global solutions and blow-up phenomena in finite time of the following *quasilinear dissipative Kirchhoff's string problem with initial conditions*

$$\begin{cases} u_{tt} - \varphi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t = f(u), & x \in \mathbb{R}^N, t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^N \end{cases}$$

where $(\varphi(x))^{-1} \coloneqq g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N),\ N \geq 3$, δ the resistance modules and f the external force.

The equation was firstly proposed by *Gustav Kirchhoff* for the study of vibrating stings (one-dimensional), membranes (two-dimensional), or elastic solids (three-dimensional) with $\delta = f = 0$, and takes the form

$$ph\frac{\vartheta^2 u}{\vartheta t^2} = \left\{p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\vartheta u}{\vartheta x}\right)^2 dx\right\}\frac{\vartheta^2 u}{\vartheta x^2}.$$

for $0 < x < L,\ t \geq 0$.

The physical interpretation of $u = u(x,t)$ is the displacement in some direction of the point x at time, $t \geq 0$, where by E we denote the *Young-modules*[12], p the mass density, h the cross-section area, L the length, and p_0 the initial axial tension. When $p_0 = 0$ the equation is considered to be of *degenerate type*, i.e., an unstretched string or its higher dimensional generalization; otherwise, it is of *non-degenerate type* and the equation models a stretched string or its higher dimensional generalization (See more in [9], [10].).

12. *Elasticity's modules* or *Young-modules* is the proportion coefficient between stress and strain (deformation) and represents the stress that causes $\varepsilon = 1$, i.e., the displacement of the string is equal to its initial length; $\Delta L = L$. We use the notation $E = \sigma/\varepsilon = FL/h\Delta L$, where *F* is the force exerted on an object under tension, *h* the actual cross-sectional area, *ΔL* the displacement of the length of the object, and *L* its initial length. Is measured in Pa ($1\ Pa = 1\ N/m^2 = 1\ kg/ms^2$). In material science we distinguish *elasticity's modules* in *tension* and *compression*.

For the purposes of this work, we have made the following arrangement: In Chapter I we introduce the *dynamical system, -the framework of analysis of evolution equation-* and give an example from classical mechanics to specify the concept of *phase space*. In Chapter II we begin the study of our problem. In the first section we make a brief review in the natural background of *hyperbolic equations with non-constant diffusion coefficient*, and give some known results concerning the study of our problem. In the next section we represent the space setting of the problem and the necessary embeddings for the continuation of our study. In the third section, we prove the existence and uniqueness of the *weak-solutions*, in the fourth section we study the existence of *global-solutions* and the energy estimates of those, using the *potential well method*. In the fifth section we end our study by giving some results concerning the *blow-up phenomena* of the solutions of our problem using the *concavity method* (see [3]).

I. Dynamical Systems

The "*space-time-condition*" as *primus* principle of validity of motion presupposes the existence of one or more elementary factors which represent *the connection of events*. Since our concepts of change are based upon this connection, defining a "framework of change", i.e., a space that represents rates of change depending on the initial data at some given time, is as a matter of fact one-way. Even though this is an arbitrary definition, this framework is actually what we usually call a *dynamical system*. Throughout this chapter we follow the mathematical description of a dynamical system, and in section 2 we will return to our preceding definition in order to explain in depth the concept of *phase space* or *state space* as it may also be found in literature.

1. Definitions, and elementary properties

In the case of continuous time[13] we have the following definition

Definition 1.1.(a) A family of maps $\{M_t\}_{t\geq 0}\colon X \to X$ such that $M_0 = I$ and

$$M_{t+s} = M_t \circ M_s,\ \ \forall t, s \geq 0$$

is called a *semi-flow*.
(b) A family of maps $\{M_t\}_{t\in\mathbb{R}}\colon X \to X$ such that $M_0 = I$ and

$$M_{t+s} = M_t \circ M_s,\ \ \forall t, s \in \mathbb{R}$$

is called a *flow*.

We say that a family of maps $\{M_t\}$ that is a *semi-flow* or a *flow* is a *dynamical system with continuous time.*

Remark. If $\{M_t\}$ is a flow, then $M_t \circ M_{-t} = M_{-t} \circ M_t = M_0 = I$, and therefore each map M_t is invertible with its inverse given by $M_t^{-1} = M_{-t}$.

13. See Chapter 2 in [6] for a description of dynamical systems for discrete and continuous time.

Proposition 1.2. Assume F: $\mathfrak{R}^N \rightarrow \mathfrak{R}^N$ is a *continuous function* such that, given $u_0 \in \mathfrak{R}^N$, the initial-value problem

$$\begin{cases} \dot{u}(t) = F(u) \\ u(0) = u_0 \end{cases}$$

has a unique solution $u(t, u_0)$ defined for $t \in \mathfrak{R}$. Therefore, the family of maps $\{M_t\}$ from $\mathfrak{R}^N$ to $\mathfrak{R}^N$ defined for each $t \in \mathfrak{R}$ by

$$M_t(u_0) = u(t, u_0)$$

is a *flow*. ■
For the proof see Proposition 2.3 in [6].

Relating this result with the definition for the semigroup of operators (see in preliminary), the semigroup $\{S(t)\}_{t\geq 0}$ defines a dynamical system in X *if and only if* the operators are continuous maps from X to X, i.e., the family of maps $\{S(t)\}_{t\geq 0}$ is a *flow*. From this point of view, we may in addition review *the concept of orbits* which represent, as we shall, see the evolution of our solutions.

Definition 1.3.(a) For a *semi-flow* $\{M_t\}_{t\geq 0}$ of X, given a point *x* in X, we define the *positive semi-orbit* of *x* by the set,

$$\gamma^+(x) = \gamma_M^+(x) = \{M_t(x): t \geq 0\}.$$

Similarly, we define for a *flow* $\{M_t\}_{t\in\mathbb{R}}$ the *negative semi-orbit* by the set

$$\gamma^-(x) = \gamma_M^-(x) = \{M_{-t}(x): t \geq 0\}.$$

Thus, the orbit of *x* is given by

$$\gamma(x) = \gamma_M(x) = \{M_t(x): t \in \mathbb{R}\}.$$

(b) In the same way, for every point *u* in X, the *positive semi-orbit* of the dynamical system, or the orbit that begins from *u* is given by the set

$$\gamma^+(u) = \bigcup_{t\geq 0} S(t)u,$$

and the *negative semi-orbit*, or the orbit that ends in *u* is given by the set

$$\gamma^-(u) = \bigcup_{t\geq 0} S(-t)u.$$

The orbit of the dynamical system defined by $\{S(t)\}_{t\geq 0}$ is obviously the *union of these semi-orbits*, i.e.,

$$\gamma(u) = \gamma^+(u) \bigcup \gamma^-(u).$$

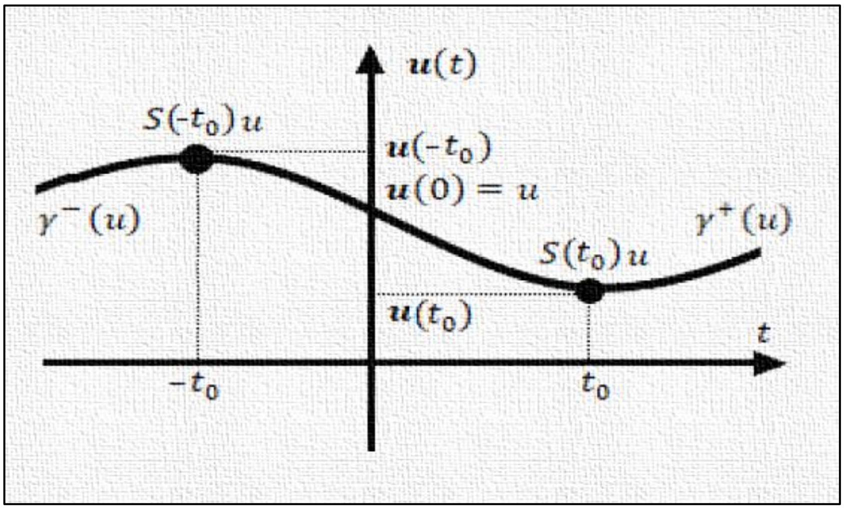

Figure 2: The orbit of a dynamical system with initial condition $\boldsymbol{u}(0) = u$.

In figure 2 the *time-axis* is defined as *strictly positive*, i.e., the *negative semi-axis* is understood as taking time with minus sign to express the *inversibility property of* $\{S(t)\}_{t\geq 0}$. Consequently, the *negative semi-orbit* is valid *if and only if* the dynamical system possesses the *backwards uniqueness property;* $S(t)$ is one-to-one.

2. Phase space

In Proposition 1.2. we assumed that our function is a continuous vector-valued function defined in $\mathfrak{R}^N$. This means that the function *u* which satisfies the initial-value problem will be a vector function, e.g., $u = (u_1, \cdots, u_N)$. We may take this vector as a row-vector, i.e., $u = (u_1, \cdots, u_N)$; or as a column-vector, i.e., $u = (u_1, \cdots, u_N)^T$ depending on the validity of our calculations.

Such a function may be considered as a parametrical representation of the N-dimensional generalization of the curve in the plane $u_1 u_2 \cdots u_{N-1} u_N$, and usually we interpret this representation as an *orbit* of a single particle in $\mathfrak{R}^N$ with velocity given by the differential equation. The plane $u_1 u_2 \cdots u_{N-1} u_N$ is regarded as the *phase space* or *state space* of our dynamical system, and a venerable collection of orbits determine the behavior of the evolution of the system.

In order to clarify in a more explicit way *the concept of phase space* we consider a single particle moving in $\mathfrak{R}^N$. Apparently, the *position*, the *velocity* and the *acceleration* of our particle will be represented by vectors in $\mathfrak{R}^N$ as we have already introduced. Therefore, we have respectively,

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \ \ \boldsymbol{u} \equiv \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{pmatrix}, \ \ \boldsymbol{a} \equiv \ddot{\boldsymbol{x}} = \begin{pmatrix} \ddot{x}_1 \\ \vdots \\ \ddot{x}_N \end{pmatrix}.$$

Under these assumptions, *Newton's second law* ($F = ma$) takes the form

$$\boldsymbol{F}\big(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)\big) = m\ddot{\boldsymbol{x}}(t),$$

where m is the *mass* of the particle, and $\boldsymbol{F}\colon \mathfrak{R}^{N} \times \mathfrak{R}^{N} \rightarrow \mathfrak{R}^{N}$, expresses the *force law*, which may depend on both the position and velocity of the particle.

Assuming that our force is *velocity-independent* we can "integrate" our second-order differential equation and receive a pair of first-order differential equations, using the velocity of our particle as a new variable. Thus, we take the following system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{u}(t) \\ \dot{\boldsymbol{u}}(t) = \dfrac{\boldsymbol{F}\big(\boldsymbol{x}(t)\big)}{m}. \end{cases}$$

If $\boldsymbol{x}(t)$ is the solution to the first equation and substitute $\dot{\boldsymbol{x}}(t)$ for $\boldsymbol{u}(t)$ in the second equation, then we see that $\boldsymbol{x}(t)$ satisfies *Newton's second law*. The pairs of the form $(\boldsymbol{x}(t), \boldsymbol{u}(t))$ that satisfy this system are regarded as the *phase space* or *state space* of the particle in $\mathfrak{R}^{N}$.

Clearly these pairs are represented in a $\mathfrak{R}^{2N}$ space with the evolution given by the equations using in addition some appropriate initial condition, usually $\boldsymbol{x}(0) = \boldsymbol{x}_0$, $\boldsymbol{u}(0) = \boldsymbol{u}_0$, where $\boldsymbol{x}_0, \boldsymbol{u}_0 \in \mathfrak{R}^{N}$ are the initial position and velocity vectors, respectively.

If in addition we consider the *Hamiltonian approach of classical mechanics* the *position-velocity* pair is replaced by the *position-momentum* pair, and therefore the above system takes the form

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \dfrac{\vartheta \mathcal{H}}{\vartheta \boldsymbol{p}} \\ \dot{\boldsymbol{p}}(t) = -\dfrac{\vartheta \mathcal{H}}{\vartheta \boldsymbol{x}}, \end{cases}$$

with

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{p}) = (2m)^{-1} \sum_{j=1}^{N} p_j^2 + V(\boldsymbol{x})$$

be the energy function (kinetic plus potential energy)[14] of the system, which is referred as the "*Hamiltonian*" of our system.

14. We write $\boldsymbol{\mathcal{K}} = \frac{1}{2}\sum_{j=1}^{N} m_j u_j^2 = \frac{1}{2m}\sum_{j=1}^{N} p_j^2$ to express the *kinetic energy* of a particle in $\mathfrak{R}^{N}$ with p_j denoting the *momentum* of the particle, and $V(\boldsymbol{x})$ the potential energy function with $-\vartheta V/\vartheta \boldsymbol{x} = \boldsymbol{F}$.

Similarly, the set of possible pairs of the form $(\boldsymbol{x}(t), \boldsymbol{p}(t))$ are regarded as the *phase space* of the particle in $\mathfrak{R}^N$, with the appropriate initial conditions to be $\boldsymbol{x}(0) = \boldsymbol{x}_0$, $\boldsymbol{p}(0) = \boldsymbol{p}_0$, where $\boldsymbol{x}_0, \boldsymbol{p}_0 \in \mathfrak{R}^N$ are the initial position and momentum vectors, respectively.

The *phase space* therefore represents all the possible states of the dynamical system which describe the evolution according to the differential system of equations and the initial conditions.

Example 2.1. Consider the motion of a single particle in $\mathfrak{R}^N$ with no force acting upon it. Therefore, we may take the potential V to be identically equal to zero. According to *Hamilton's equations*, then, we have

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \dfrac{\vartheta \mathcal{H}}{\vartheta \boldsymbol{p}} = \dfrac{\vartheta}{\vartheta \boldsymbol{p}} \left\{ (2m)^{-1} \sum_{j=1}^{N} p_j^2 \right\} = (m)^{-1} \sum_{j=1}^{N} p_j = (m)^{-1} \boldsymbol{p} \\ \dot{\boldsymbol{p}}(t) = -\dfrac{\vartheta \mathcal{H}}{\vartheta \boldsymbol{x}} = 0. \end{cases}$$

This means that the momentum of the particle is *independent of time*, which indicates that the position of the particle is *linear in time*. Explicitly we write

$$\frac{dx_j}{dt} = (m)^{-1} p_j \Rightarrow dx_j = (m)^{-1} p_j dt \Rightarrow \int dx_j = \int (m)^{-1} p_j dt + c_j \Rightarrow$$
$$x_j = (m)^{-1} p_j t + c_j,$$

and taking the initial conditions $x_j(0) = x_0, p_j(0) = p_0$ we have

$$\begin{cases} x_j(t) = (m)^{-1} p_0 t + x_0 \\ p_j(t) = p_0. \end{cases}$$

Expressing the solution pair $\left(x_j(t), p_j(t)\right)$ using matrix-notation we get

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \frac{1}{m} \begin{bmatrix} p_1(0) \\ \vdots \\ p_N(0) \end{bmatrix} t + \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix}$$

$$\begin{bmatrix} p_1(t) \\ \vdots \\ p_N(t) \end{bmatrix} = \begin{bmatrix} p_1(0) \\ \vdots \\ p_N(0) \end{bmatrix}.$$

To simplify the representation of our phase space we can use *conserved quantities* or *constant of motions* as may also be found in literature. The significance of those lies in the fact that if we are able to determine them, then each solution of the system must lie entirely in the level surface of the defined conserved quantity. For instance, instead of representing the above solution pair in the original phase space $\mathfrak{R}^{2N}$ we can use the conservation of the Hamiltonian $\mathcal{H}$ and of the angular momentum $\mathcal{J}$ to represent the trajectories inside the joint level sets of $\mathcal{H}$ and $\mathcal{J}$. Obviously, the simplification depends on the determination of the conserved quantities and by extension upon the dimensionality of the joint level sets of them.

For a further reading see Chapter 2 in [1], and Chapter 9 in [14].

II. Existence of global solutions

In this chapter we will study the solvability of a hyperbolic wave type problem with initial conditions in $\mathfrak{R}^N$. In the first section we review the physical background of the problem and give some known results concerning the global solutions and blow-up phenomena. The functional setting of the problem takes place in the second section where as we shall see the energy space $\mathcal{X}_0 \coloneqq D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ introduced to overcome the non-compactness of the operators which arise in unbounded domains. In the third and fourth section we prove the existence and uniqueness of global solutions and the energy estimates of those. In the fifth section we complete our study with the blow-up phenomena analysis of the problem.

1. The equation and some known results

In various areas in mathematical physics the study of wave phenomena is connected with the study of equations of the following form[15]

$$u_{tt} - \Delta u = f(u),$$

where $u = u(x,t)$ is the unknown, the Laplacian operator Δ is taken with respect to the spatial variables, and the real-valued function f defined in the space of the unknown function represents the external force. The above equation is referred as the *nonhomogeneous wave equation* in literature and modifications of this equation give rise to further studies of wave phenomena.
In this chapter we will study a modification of the nonhomogeneous wave equation which is referred as the *quasilinear dissipative Kirchhoff's type problem with initial conditions* and takes the following form

$$\begin{cases} u_{tt} - \varphi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t = f(u), \quad x \in \mathbb{R}^N, t \geq 0 & (1.1) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N & (1.2) \end{cases}$$

where $u = u(x,t)$ is the unknown, $\delta > 0$ the damping term, $\varphi(x)$ is the *non-constant diffusion coefficient* which represents the wave propagation in nonhomogeneous medium (changeable density), and f is the external force which takes the *subcritical power nonlinearity form*; $f(u) = |u|^2 u$.

15. See more in (Section 2.4 and Chapter 12 in [5]).

For a widely description of physical phenomena that lead to relative mathematical problems of the above type we refer to [7], [8], [9], [10], and for a further reading we recommend the references of the paper [10], mentioned in Chapter 3. Indicatively we mention the *Ginzburg-Landau theory*[16], where in the case of a nonhomogeneous superconductor the diffusion coefficient -which represents the coherence length of the superconductive electrons- is considered to be non-constant with respect to spatial variable.

16. The first observations regarding the effacing of the electric resistance of pure metals at very low temperatures, which set forth the emergence of the superconductivity phenomenon, were made by H. Kamerlingh Onnes in the paper "***The liquefication of helium***" (KNAW, Proceedings, 11, 1908-1909, Amsterdam, 1909, pp. 168-185) followed by the paper series "***Further experiments with liquid helium***" (KNAW, Proceedings, 1910-1923). On the next decade, and more precisely in 1933, Meissner W. and Ochsenfeld R. published a paper in *Die Naturwissenschaften* **21** 787, under the title, "***A new effect concerning the onset of superconductivity***" ("***Eh neuer Effekt bei Eintritt der Supraleitfiihigkeit***"). According to this if one places a cylindrical superconductor, e.g., lead or tin, below its transition temperature in a uniform magnetic field perpendicular to the cylinder axis the field-line pattern in the region outside the superconductor changes almost to that which would be expected if the permeability of the superconductor was zero, or the diamagnetic susceptibility was -1/(4π), which contradicts the views of "frozen in" magnetic fields in superconductors (See ***Meissner and Ochsenfeld revisited*** by Allister M. Forrest; Department of Physics, Paisley College of Technology, Paisley, Renfrewshire, Scotland; Received July 1983). The theoretical description of the electromagnetic field in a superconductor was given the following year, by F. and H. London in their paper, "***The Electromagnetic Equations of the Supraconductors***" (Clarendon Laboratory, Oxford, communicated by F. A. Lindemann, F.R.S.; Received October 23, 1934), which was consistent with Meissner's effect, and clarified the dependance of the superconducting current with the field. In comparison with the electromagnetic effects in a superconductor, the thermoelectric effects were still a living problem, and only 16 years later and specifically in 1950 London's theory generalized to overcome the difficulties regarding its application in stronger magnetic fields, the negative values of the surface energy at the interface between normal and superconducting phases etc. The *Ψ-theory of superconductivity* or the *Ginzburg-Landau theory* as may also be found in literature was the answer to this generalization; the basis of this was the preceding theory on phase transitions proposed by L. Landau, and the paper on superfluidity (May 15, 1941) which referred to superconductivity as the superfluidity of electron liquid in metals ("***On Superconductivity and Superfluidity***", Nobel Lecture, December 8, 2003; by Vitaly L. Ginsburg, P. N. Lebedev Physics Institute, Russian Academy of Sciences, Moscow, Russia). The quantum approach of the theory was already proposed by F. London and as *Ginsburg-Landau theory* was quasi-macroscopic, the microscopic theory of superconductivity was the next link of the chain. This theory was given in 1957 and is widely known as the BCS theory ("***Theory of Superconductivity***" by J. Bardeen, L. N. Cooper, and J. R. Schrieffer; Dept. of Physics, University of Illinois, Urbana, Illinois; Physical Review, Vol. 108, N. 5; December 1, 1957). Many efforts have been made since the advent of the BCS theory on the radical elevation of

Concerning the solvability and blow-up phenomena of the Cauchy-type problem for nonlinear hyperbolic equations in papers [7], and [8] we see the study of the global existence, the blow-up and the asymptotic behavior of the solutions of quasilinear wave equations with weak damping in $\mathfrak{R}^N$. Specifically, in [8] we see that under certain assumptions on the initial data, solutions exist globally in the energy space $\mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$. In this paper the existence of a weak solution to the problem is obtained using the *Faedo-Galerkin approximation* (see [8]) where the use of the *Banach fixed-point theorem* is used to obtain the uniqueness of the solution. The global existence is proved using the *method of modified potential well* (we shall see this method explicitly in the following sections) and the proof of *blow-up of solutions in finite time* is given on the consideration of *negative initial energy*. For a widely description of some known results, see paper [10].

2. Functional analysis of the problem

In the study of problems of the type $(1.1), (1.2)$ the functional importance of the differential operator $-\varphi(x)\Delta$ and the asymptotic behavior of the diffusion coefficient -which depends on the equation that describes the physical phenomenon- constitute the *primus axes* of the solvability procedure.

In this case, we consider the non-constant coefficient $\varphi(x)$ with the following form

$$\varphi(x) = c_0 + \sum_{k=1}^{\infty} c_k(\varepsilon|x|^k), \;\; \varepsilon > 0,$$

where if we assume that ε is sufficiently small, then $\varphi(x)$ represents the slowly varying wave velocity around the velocity c_0. These kinds of bounded functions g, where $g(x) \equiv (\varphi(x))^{-1} \to 0$, as $|x| \to \infty$ -with slow rate-, it could be considered to belong in a Lebesgue space of the type $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, for some $p > 0$. More precisely we assume that the function $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ satisfies the following condition

$$(\mathfrak{G}) \qquad \varphi(x) > 0, \;\; \forall x \in \mathbb{R}^N \; and \; (\varphi(x))^{-1} \coloneqq g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

For the study of the problem $(1.1), (1.2)$ as a dynamical system we introduce the phase space $\mathcal{X}_0 \coloneqq \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathcal{D}(A)$, where the space $\mathcal{D}^{1/2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the "energy norm"

$$\|u\|^2_{\mathcal{D}^{1,2}} \coloneqq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.$$

the critical temperature and the construction of high and room-temperature superconductors (HTSC and RTSC) but the main objection against these efforts is the *crystal stability condition* (see "***On Superconductivity and Superfluidity***", Nobel Lecture, December 8, 2003; by Vitaly L. Ginsburg, P. N. Lebedev Physics Institute, Russian Academy of Sciences, Moscow, Russia).

It is well known[17] that $\mathcal{D}^{1,2}(\mathbb{R}^N) := \{u \in L^{2N/(N-2)}(\mathbb{R}^N):\ \nabla u \in (L^2(\mathbb{R}^N))^N\}$ and that $\mathcal{D}^{1,2}$ is embedded continuously in $L^{2N/(N-2)}$, i.e., $\exists k > 0$ such that

$$\|u\|_{\frac{2N}{N-2}} \leq k\|u\|_{\mathcal{D}^{1,2}} \tag{2.1}$$

The space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is also a *separable Hilbert space* (see Def. 1.1.6 in Preliminary) equipped with the inner product

$$(u,v)_{\mathcal{D}^{1,2}} := \int_{\mathbb{R}^N} \nabla u \nabla v\, dx.$$

Remark 2.1 (a) In the case of a bounded domain Ω, we have the relation

$$\mathcal{D}^{1,2}(\Omega) \equiv H_0^1(\Omega) \stackrel{\text{def}}{=} \left\{u \in L_0^2(\Omega):\ \frac{\vartheta u}{\vartheta x_i} \in L_0^2(\Omega),\ \ i = 1,\cdots,N\right\}.$$

(b) In the case of an unbounded domain, i.e., a domain with infinite volume, or the whole $\mathbb{R}^N$, the following embedding occurs

$$H^1(\mathbb{R}^N) \subset \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Before we proceed with the analysis of the problem, we shall give the following generalized version of Poincare's inequality (see Remark on pp.12).

Lemma 2.2. Assume $g \in L^{N/2}(\mathbb{R}^N)$. Then there exist $\xi > 0$, such that

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq \xi \int_{\mathbb{R}^N} g u^2\, dx$$

for every $u \in C_0^\infty(\mathbb{R}^N)$.

Proof. Since $g \in L^{N/2}(\mathbb{R}^N)$ and $u \in C_0^\infty(\mathbb{R}^N)$, we can use the *Hölder inequality* (see pp. 9; inequality [e]) with $p = \frac{N}{2}$ and $q = \frac{2N}{N-2}$ (where the q-factor is valid from the definition of the $\mathcal{D}^{1,2}(\mathbb{R}^N)$). This leads to

$$\int_{\mathbb{R}^N} |g|u^2\, dx \leq \left\{\int_{\mathbb{R}^N} |g|^{N/2}\, dx\right\}^{\frac{2}{N}} \left\{\int_{\mathbb{R}^N} u^{2N/(N-2)}\, dx\right\}^{\frac{N-2}{N}} = \|g\|_{N/2}\|u\|_{2N/(N-2)}^2.$$

17. See Reference_67 of the paper [10], mentioned in section 3.2. We also recommend the paper "*New estimates for the steady-state Stokes problem in exterior domains with applications to the Navier-Stokes problem*", by G.P. Galdi, and C.G. Simader (Differential and Integral Equations, Vol. 7, N. 3, May 1994, pp. 847-861); where we see that for $q \in (1,\infty)$ the *homogeneous* Sobolev space $\mathcal{D}_0^{1,q}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the semi-norm $|u|_{1,q,\Omega} \equiv \left(\int_\Omega |\nabla u|^q\right)^{1/q}$.

From the embedding inequality (2.1), we obtain

$$\int_{\mathbb{R}^N} |g|u^2\, dx \le k^2 \|g\|_{N/2} \|u\|^2_{\mathcal{D}^{1,2}}.$$

Therefore, we can easily find that, $\xi = k^{-2}\|g\|^{-1}_{N/2}$. Q.E.D.

An elementary step for the continuation of our study are the *compact embeddings* (see pp. 8, Def. 2.1.2.) where as we shall see elucidate the connection between the spaces of our problem, and constitute the pillars of the *evolution-triple* (see Theorem 2.4).

Lemma 2.3. Assume $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then the embedding $\mathcal{D}^{1,2} \subset L^2_g$ is compact.

Proof. From the Def. 2.1.2 we know that the space $\mathcal{D}^{1,2}$ is *compactly embedded* in L^2_g *if and only if*,

(a) $\|u\|_{L^2_g} \le C\, \|u\|_{\mathcal{D}^{1,2}}$ for every $u \in \mathcal{D}^{1,2}$ and $C > 0$,

and

(b) each bounded sequence in $\mathcal{D}^{1,2}$ is *pre-compact* in L^2_g, *i.e.*, if $\{u_j\}_{j=1}^\infty$ is a sequence in $\mathcal{D}^{1,2}$ with $sup_j \|u_j\|_{\mathcal{D}^{1,2}} < \infty$, then there exists some subsequence $\{u_{j,\ell}\}_{\ell=1}^\infty \subseteq \{u_j\}_{j=1}^\infty$which converges to some limit u in L^2_g; this means that, $\lim_{\ell\to\infty} \|u_{j,\ell} - u\|_{L^2_g} = 0$.

The first condition is satisfied since we already have seen (Lemma 2.2) that $\|u\|^2_{L^2_g} \le C\|u\|^2_{\mathcal{D}^{1,2}}$ with $C = k^2\|g\|_{N/2}$. To prove the second condition, we need to show that a bounded sequence of $\mathcal{D}^{1,2}$ is a *Cauchy sequence*[18] in L^2_g, and use the property that in finite dimensional space any Cauchy sequence is convergent[19].

For this purpose, we assume that $\{u_n\}$ is the bounded sequence of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. For all the positive integers m and n we have

$$\int_{\mathbb{R}^N} g(u_n^2 - u_m^2)\, dx = \int_{\mathbb{R}^N} g(u_n - u_m)(u_n + u_m)\, dx,$$

and by *Hölder inequality* for $p = \frac{2N}{N+2}$ and $q = \frac{2N}{N-2}$ we obtain

$$\int_{\mathbb{R}^N} g(u_n^2 - u_m^2)\, dx \le \|g(u_n - u_m)\|_{2N/(N+2)}\, \|u_n + u_m\|_{2N/(N-2)}.$$

From the embedding inequality (2.1) we derive

18. See [Def. 1.1.3(a); pp. 2].

19. See [4] pp. 14.

$$\int_{\mathbb{R}^N} g(u_n^2 - u_m^2)\,dx \leq k\|g(u_n - u_m)\|_{2N/(N+2)}\,\|u_n + u_m\|_{\mathcal{D}^{1,2}},$$

and from the generalized version of Poincare's inequality (Lemma 2.2) we get

$$\int_{\mathbb{R}^N} g(u_n^2 - u_m^2)\,dx \leq \mathscr{k}\|g(u_n - u_m)\|_{2N/(N+2)},$$

where $\mathscr{k} = \tilde{k}^{-1}\|g\|_{N/2}^{-1/2}\|u_n + u_m\|_{L_g^2}$, and the constants are considered in a generic sense.

Since $\{u_n\}$ is a bounded sequence of $\mathcal{D}^{1,2}(\mathbb{R}^N)$, by Remark 2.1 (a) we have that $\{u_n\}$ is also a bounded sequence in $H_0^1(B_R)$, where B_R is any open ball of $\mathbb{R}^N$ with center 0 and radius *R*. Therefore, by the *classical Sobolev embeddings*[20] we conclude that $\{u_n\}$ has a convergent subsequence in $L^2(B_R)$; consequently, and in $L^{2N/(N+2)}(B_R)$. In continuation, following a *diagonalization process*[21] we can find a subsequence (for convenient we shall use the same notation $\{u_n\}$), which converges in $L^{2N/(N+2)}(B_R)$, for each $R > 0$.

Assume ε a strictly-positive number, *i.e.*, $\varepsilon > 0$. Then we have

$$\|g(u_n - u_m)\|_{2N/(N+2)} = \left\{\int_{|x|\leq R} |g(u_n - u_m)|^{\frac{2N}{N+2}}\,dx + \int_{|x|>R} |g(u_n - u_m)|^{\frac{2N}{N+2}}\,dx\right\}^{\frac{N+2}{2N}}$$

$$\leq \|g(u_n - u_m)\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N - B_R)} + \|g(u_n - u_m)\|_{L^{\frac{2N}{N+2}}(B_R)}$$

For the first integral we have that

$$\|g(u_n - u_m)\|_{L^{2N/(N+2)}(\mathbb{R}^N - B_R)} \leq \|g\|_{L^{N/2}(\mathbb{R}^N - B_R)}\|u_n - u_m\|_{L^{N/2}(\mathbb{R}^N - B_R)}.$$

Since $\{u_n\}$ is a bounded sequence of $\mathcal{D}^{1,2}$ and $g \in L^{N/2}$, we can choose a R_0 sufficiently large, such that

$$\|g(u_n - u_m)\|_{L^{2N/(N+2)}(\mathbb{R}^N - B_R)} \leq \varepsilon/2, \quad \forall m, n, \ \ if \ \ R > R_0.$$

For the second integral we obtain

$$\|g(u_n - u_m)\|_{L^{2N/(N+2)}(B_{R_0})} \leq \|g\|_{L^\infty(B_{R_0})}\|u_n - u_m\|_{L^{2N/(N+2)}(B_{R_0})} < \varepsilon/2$$

20. See Section 2.3 in Preliminary.

21. See Gray, Robert (1994), "*Georg Cantor and Transcendental Numbers*", American Mathematical Monthly, 101 (9): 819–832. For a further reading on the theory of transfinite numbers see "*Contributions to the Founding of the Theory of Transfinite Numbers*" (Dover Books on Mathematics) -1st Edition- by Georg Cantor, Philip Jourdain (Translator); Published 1955.

under the presupposition that m and n are sufficiently large. Therefore, $\{u_n\}$ is a *Cauchy sequence* in $L_g^2(\mathbb{R}^N)$. Q.E.D.

In further examination of the operator $-\varphi\Delta$, we observe that its *symmetry property* with respect to the inner product of $L^2(\mathbb{R}^N)$ is not satisfied, *i.e.*,

$$(-\varphi\Delta u, v)_{L^2} \neq (u, -\varphi\Delta v)_{L^2} \ \ \textit{for all} \ \ u, v \in D(-\varphi\Delta),$$

where $D(-\varphi\Delta)$ is the domain of the operator $-\varphi\Delta$.

To overcome this difficulty[22], we have to study the operator in a weighted space $(L_g^2(\equiv L_{\varphi^{-1}}^2))$. Following this, is the analysis of the *weight*-$\varphi(x)$, and more precisely its boundness characteristics, since if we can clarify its behavior as $|x| \to \infty$, we can define the physical functional environment of the problems in which such *weighted-functions* take place.

More explicitly, if the weight $\varphi(x)$ is of the form

$$0 < c_1 \leq \varphi(x) \leq c_2,$$

then it is obvious, that the respective levels are equivalent. Relevant is the case, when the problem is studied in bounded domains, independently from the boundness characteristics of the function $\varphi(x)$. Especially, in the case where the space is the $\mathbb{R}^N$ and the diffusion coefficient $\varphi(x) \to \infty$, as $|x| \to \infty$, the conclusions are different.

Following the *Friedrichs' Extension Theorem* (see Theorem 2.4.) for the studied operator we shall prove that the functional analysis of the problem $(1.1), (1.2)$ takes place in the space $\mathcal{X}_0$.

Theorem 2.4. Assume the operator $A_0: D(A_0) \subseteq X \longrightarrow X$, where $dimX = \infty$ and $\overline{D(A)} = X$, is *symmetric* in the (real) Hilbert space X and that the operator A_0 is *strongly monotone*, i.e.,

$$(A_0 u, u)_X \geq c\|u\|_X^2, \ \ \forall u \in D(A_0)$$

and where $c > 0$.

Then there exists a *self-adjoint extension* $A: D(A) \subseteq X_E \subseteq X \to X$ of the operator A_0 where X_E is the energetic space of A_0, which satisfies the following conditions

(1) the operator is *strongly monotone*, *i.e.*,

$$(Au, u)_X \geq c\|u\|_X^2, \ \ \forall u \in D(A),$$

(2) the inverse operator $A^{-1}: X \to X$ exists and is linear, continuous and self-adjoint. This means that the equation

$$Au = f, \quad u \in D(A), \quad f \in X,$$

has the unique solution; $u = A^{-1}f$,

22. See pp. 44 in [10].

(3) the operator $A^{-1}: X \to X_E$ is linear and continuous,
(4) the embeddings $X_E \subset X \subset X_E^*$ are continuous,
(5) the operator A has the extension $A_E: X_E \to X_E^*$, where A_E is the dual representation of X_E, *i.e.*, A_E is a *homeomorphism* and

$$\langle A_E u, u\rangle = \|u\|_E^2, \ \ \forall u \in X_E.$$

We also have that

$$A^{-1} f = A_E^{-1} f, \ \ \forall u \in X,$$

(6) if the embedding $X_E \subset X$ is *compact*, then the operator $A^{-1}: X \to X$ is also compact. ■

Let us consider the equation

$$-\varphi(x)\Delta u(x) = \eta(x), \quad x \in \mathbb{R}^N, \tag{2.2}$$

without boundary condition. Obvious for every $u, v \in C_0^\infty(\mathbb{R}^N)$, we have

$$(-\varphi\Delta u, v)_{L_g^2} = -\int_{\mathbb{R}^N} g\varphi\Delta uv\, dx = -\int_{\mathbb{R}^N} \Delta uv\, dx$$

and using *Green's second formula* and the *compact support of* u, v we obtain

$$(-\varphi\Delta u, v)_{L_g^2} = \int_{\mathbb{R}^N} \nabla u \nabla v\, dx. \tag{2.3}$$

By the definition of the space $L_g^2(\mathbb{R}^N)$ and (2.3) it is natural to consider the equation (1.4), as an operator equation

$$A_0 u = \eta, \ \ A_0: D(A_0) \subseteq L_g^2(\mathbb{R}^N) \to L_g^2(\mathbb{R}^N), \tag{2.4}$$

where $A_0 \coloneqq -\varphi\Delta$ with domain of definition $D(A_0) = C_0^\infty(\mathbb{R}^N)$, and $\eta \in L_g^2(\mathbb{R}^N)$. Relation (2.3) implies that the operator A_0 is symmetric with respect to the inner product of the weighted space L_g^2 and not symmetric in the standard *Lebesgue space* L^2. From Lemma 2.2. and equation (2.3) we have

$$(A_0 u, u)_{L_g^2} = \int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq \xi \int_{\mathbb{R}^N} g u^2\, dx$$

or

$$(A_0 u, u)_{L_g^2} \geq \xi \|u\|_{L_g^2}^2, \quad \forall u \in D(A_0), \tag{2.5}$$

where $\xi > 0$ is the constant fixed in Lemma 2.2, *i.e.*, the operator A_0 is *strongly monotone*.

Therefore, the assumptions for the *Friedrichs' Extension Theorem* (see Theorem 2.4.) are satisfied. Consequently, we can define the *energetic inner product* given by the equation (2.3) as follows

$$(u,v)_E = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx.$$

The energetic space X_E is defined as the completion of $D(A_0)$ with respect to the product $(u,v)_E$, *i.e.*, the energetic space coincides with the *homogeneous Sobolev space* $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The energetic extension of the operator A_0

$$A_E \coloneqq -\varphi\Delta \colon \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathcal{D}^{-1,2}(\mathbb{R}^N),$$

is defined as the duality mapping of $\mathcal{D}^{1,2}(\mathbb{R}^N)$, and according to the Theorem 2.4, for every $\eta \in \mathcal{D}^{-1,2}(\mathbb{R}^N)$, the equation (2.2) has a unique solution. We also define the set $D(A)$ as the set of all the solutions u of the equation

$$A_E u = \eta, \;\; \eta \in L^2_g(\mathbb{R}^N).$$

Hence, the *Friedrichs' extension* A of A_0 is defined as the restriction of the energetic extension A_E to the set $D(A)$. The operator A is self-adjoint[23] and therefore graph-closed[24,25]. This implies that the set $D(A)$ is a Hilbert space with respect to the graph inner product

$$(u,v)_{D(A)} \coloneqq (u,v)_{L^2_g} + (Au,Av)_{L^2_g},$$

for every $u,v \in D(A)$.

The norm induced by the inner product $(u,v)_{D(A)}$ is

$$\|u\|_{D(A)} \coloneqq \left\{ \int_{\mathbb{R}^N} g|u|^2 \, dx + \int_{\mathbb{R}^N} \varphi|\Delta u|^2 \, dx \right\}^{1/2},$$

which is equivalent to the norm[25]

$$\|Au\|_{L^2_g} \coloneqq \left\{ \int_{\mathbb{R}^N} \varphi|\Delta u|^2 \, dx \right\}^{1/2}.$$

The *weak formulation*[26] of the equation (2.2) is

23. See [Section 2.6; pp. 85-90] in [4].
24. (Closed Graph Theorem). Let $A\colon X \to Y$, where X, Y are Banach spaces be a closed, linear operator. Then A is bounded.
25. See pp. 46 in [10].
26. See pp. 27.

$$\int_{\mathbb{R}^N} \nabla u \nabla v\, dx = \int_{\mathbb{R}^N} g\eta v\, dx,$$

where $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, for each $u \in C_0^\infty(\mathbb{R}^N)$.

From Lemma 2.3 and the results (4) and (6) of Theorem 2.4, it turns out that the embeddings

$$D(A) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^2(\mathbb{R}^N) \subset \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.6}$$

are compact and dense. Henceforth, the embedding relations (2.6) define an *evolution-quadruple* which forms the basis for the further study of the problem $(1.1), (1.2)$.

For the *general eigenvalue problem* of the *Friedrichs' extension* operator A we may derive usefull results using the continuous embeddings of (2.6) and the condition (6) of Theorem 2.4 (see pp. 47 in [10]). More precisely, for the eigenvalue problem

$$-\varphi(x)\Delta u = \mu u, \;\; x \in \mathbb{R}^N, \tag{2.7}$$

there exists a complete system of *eigen-solutions* $\{w_n, \mu_n\}$ satisfying the following relations

$$\begin{cases} -\varphi\Delta w_j = \mu_j w_j, \;\; j = 1, 2, \dots, \;\; w_j \in D(A), \\ \;0 < \mu_1 \leq \mu_2 \leq \cdots, \;\; \mu_j \to \infty, \;\; as\; j \to \infty. \end{cases}$$

The eigenfunctions $w_j,\; j = 1, 2, \dots$, belong of course in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and are also eigen-solutions of the *weak*-eigenvalue problem

$$\int_{\mathbb{R}^N} \nabla u \nabla v\, dx = \mu \int_{\mathbb{R}^N} guv\, dx, \;\; v \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

for each $u \in C_0^\infty(\mathbb{R}^N)$. We also have to note that, the eigenfunctions $w_j,\; j = 1, 2, \dots$, constitute a complete orthonormal system for the space $L_g^2(\mathbb{R}^N)$. For information regarding the asymptotic behavior of the solution u of the problem (2.7) we refer to pp. 47 in [10]; where it is mentioned that under specific arguments it can be proved that every solution of the eigenvalue-problem (2.7) converges to zero as $|x| \to \infty$.

For the *positive*[27] *self-adjoint* operator $A = -\varphi\Delta$, we can define the powers of operators as follows: for every $s \in \mathbb{R}$, the operator A^s is an *unbounded* strictly-positive operator, *self-adjoint* in the space L_g^2, with domain of definition the set $D(A^s)$, which is a dense subset of L_g^2.

27. An operator A is said to be *positive* if for all $\psi \in D(A)$, with $\psi \neq 0$, the following inequality holds

$$(\psi, A\psi) > 0.$$

The space $D(A^s)$, is also a Hilbert space with respect to the inner product

$$(u, v)_{D(A^s)} := (A^s u, A^s v)_{L_g^2}.$$

To define a proper relation between the spaces of the evolution-quadruple (2.6) and the domains of the operator A^s (for every $s \in \mathbb{R}$), we may use the notation $V_{2s} = D(A^s)$, with the following identities

$$\begin{cases} V_{-1/2} = D\left(A^{-1/2}\right) = \mathcal{D}^{-1,2}(\mathbb{R}^N) \\ V_0 = D(A^0) = L_g^2(\mathbb{R}^N) \\ V_1 = D\left(A^{1/2}\right) = \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases}$$

where for every $s_1, s_2 \in \mathbb{R}$, with $s_1 > s_2$, the embeddings $D(A^{s_1}) \subset D(A^{s_2})$ are compact. For a further reading regarding the powers of operators we refer to pp. 48 in [10] or the paper [7].

Having determined the functional background of the problem (1.1), (1.2), we are able to give the definition of the *weak-solution* for the problem, using the *evolution-quadruple* (2.6).

Definition 2.5. A *weak-solution* of the problem (1.1), (1.2), is a function $u(x, t)$ such that

(i) $u \in L^2[0, T; D(A)]$, $u_t \in L^2[0, T; \mathcal{D}^{1,2}(\mathbb{R}^N)]$, $u_{tt} \in L^2\left[0, T; L_g^2(\mathbb{R}^N)\right]$,

(ii) for every function $v \in C_0^\infty([0, T] \times (\mathbb{R}^N))$, satisfies the generalized equation

$$\int_0^T \left(u_{tt}(\tau), v(\tau)\right)_{L_g^2} d\tau + \int_0^T \left(\|\nabla u(t)\|^2 \int_{\mathbb{R}^N} \nabla u(\tau) \nabla v(\tau)\, dx\, d\tau \right)$$
$$+\delta \int_0^T \left(u_t(\tau), v(\tau)\right)_{L_g^2} d\tau - \int_0^T \left(f\left(u(\tau)\right), v(\tau)\right)_{L_g^2} d\tau = 0, \qquad (2.8)$$

where we have that; $f(s) = |s|^2 s$, and

(iii) satisfies the initial conditions

$$u(x, 0) = u_0(x) \in D(A), \quad u_t(x, 0) = u_1(x) \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Remark 2.6. Using proper density arguments, we may prove that the generalized equation (2.8) is satisfied for every $v \in L^2[0, T; \mathcal{D}^{1,2}(\mathbb{R}^N)]$. By the compactness and density of the embeddings in the *evolution-quadruple* (2.6), we have that, as in [10, Remark 3.2.5; pp. 48], the above Definition 2.5 of weak solutions implies that

$$u \in C[0, T; \mathcal{D}^{1,2}(\mathbb{R}^N)] \text{ and } u_t \in C\left[0, T; L_g^2(\mathbb{R}^N)\right].$$

Remark 2.7. Although the definition of the weak solution requires the external action to belong in the weighted space $L_g^2(\mathbb{R}^N)$, this claim is not restrictive. Indeed, by the definition of the weighted space $L_g^2(\mathbb{R}^N)$ and the condition $(\mathfrak{G})$ results that

$$L^2(\mathbb{R}^N) \subset L_g^2(\mathbb{R}^N). \tag{2.9}$$

The relation (2.9) in combination with the Remark 2.1 (b) indicate, that the solvability of the problem can be obtained in a widely class of external forces.

3. Local solution and estimates

Before we proceed with the proof of the existence of the local solution, we shall give some additional information regarding the functional spaces of the problem.

Lemma 3.1. Assume that $g \in L^{\frac{2N}{2N-pN+2p}}(\mathbb{R}^N)$. Then the following continuous embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^p(\mathbb{R}^N)$ is valid, for every $1 \leq p \leq 2N/(N-2)$.

Proof. By Definition 2.1.1 (pp. 8) we know that the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^p(\mathbb{R}^N)$ is valid *if and only if*

$$\|u\|_{L_g^p} \leq C\, \|u\|_{\mathcal{D}^{1,2}} \text{ for every } u \in \mathcal{D}^{1,2} \text{ and } C > 0.$$

Using *Hölder inequality* with $\alpha = \frac{2N}{2N-pN+2p}$ and $\beta = \frac{2N}{(N-2)p}$ we derive

$$\|u\|_{L_g^p}^p := \int_{\mathbb{R}^N} g u^p \, dx \leq \left(\int_{\mathbb{R}^N} g^\alpha \, dx \right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}^N} |u|^{p\beta} \, dx \right)^{\frac{1}{\beta}}$$

$$\leq \left(\int_{\mathbb{R}^N} g^\alpha \, dx \right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{p}{2}},$$

where in the last inequality we have used the inequality (2.1). Q.E.D.

Remark 3.2. The assumption of Lemma 3.1 is satisfied under the hypothesis $(\mathfrak{G})$, if $p \geq 2$.

Lemma 3.3. Let g satisfy condition $(\mathfrak{G})$. If $1 \leq q < p < \tilde{p} = 2N/(N-2)$, then the following weighted inequality

$$\|u\|_{L_g^p} \leq C_0 \|u\|_{L_g^q}^{1-\theta} \|u\|_{\mathcal{D}^{1,2}}^{\theta}, \tag{3.1}$$

is valid, for every $\theta \in (0,1)$, for which $1/p = \frac{1-\theta}{q} + \frac{\theta}{\tilde{p}}$ and $C_0 = k^{\theta}$.

Proof. Using the weighted interpolation inequality

$$\|u\|_{L_g^p} \leq \|u\|_{L_g^q}^{1-\theta} \|u\|_{L_g^{\tilde{p}}}^{\theta}, \text{ (See, pp. 49 in [10])}$$

and the inequality (2.1) we obtain

$$\left(\int_{\mathbb{R}^N} g|u|^p dx\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^N} g|u|^q\, dx\right)^{\frac{1-\theta}{q}} \left(\int_{\mathbb{R}^N} g|u|^{\tilde{p}}\, dx\right)^{\frac{\theta}{\tilde{p}}}$$

$$\leq \left(\int_{\mathbb{R}^N} g|u|^q\, dx\right)^{\frac{1-\theta}{q}} \left(\int_{\mathbb{R}^N} g|u|^{\frac{2N}{N-2}}\, dx\right)^{\frac{(N-2)\theta}{2N}}$$

$$= \|u\|_{L_g^q}^{1-\theta} \|u\|_{\frac{2N}{(N-2)}}^{\theta} \leq k^{\theta} \|u\|_{L_g^q}^{1-\theta} \|u\|_{\mathcal{D}^{1,2}}^{\theta}.$$

Therefore, $\|u\|_{L_g^p} \leq k^{\theta} \|u\|_{L_g^q}^{1-\theta} \|u\|_{\mathcal{D}^{1,2}}^{\theta}$, where $C_0 = k^{\theta}$. Q.E.D.

Lemma 3.4. Assume $g \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$. Then the following continuous embeddings $L_g^p(\mathbb{R}^N) \subset L_g^q(\mathbb{R}^N)$ are valid, for every $1 \leq q \leq p < \infty$.

Proof. Using *Hölder inequality* we derive

$$\int_{\mathbb{R}^N} g|u|^q\, dx \leq \left(\int_{\mathbb{R}^N} (g^{\sigma})^{\alpha}\, dx\right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}^N} (g^{\tau}|u|^q)^{\beta}\, dx\right)^{\frac{1}{\beta}},$$

where by the *Hölder-exponentials relation*[28], we have for the constants the values $\alpha = p/(p-q)$ and $\beta = p/q$. Thereby for $\sigma = (p-q)/p$ and $\tau = q/p$ we obtain the embedding inequality $\|u\|_{L_g^q} \leq \tilde{C}\|u\|_{L_g^p}$, with the constant $\tilde{C} = \|g\|_1^{(p-q)/pq}$. Q.E.D.

28. Recall that according to *Hölder inequality* (see pp. 9) the exponentials p, q, where $1 \leq p$ and $q \leq \infty$ must satisfy the condition below

$$\frac{1}{p} + \frac{1}{q} = 1.$$

so that the inequality to be valid.

In order to obtain a local existence result for the problem $(1.1), (1.2)$, we need information concerning the solvability of the corresponding nonhomogeneous linearized problem around the function v, where $(v, v_t) \in C(0,T; D(A) \times \mathcal{D}^{1,2})$ is given, restricted in the sphere B_R:

$$
(3.2) \qquad \begin{cases} u_{tt} - \varphi(x)\|\nabla v(t)\|^2 \Delta u + \delta u_t = |v|^a v, \ \ (x,t) \in B_R \times (0,T), \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \ x \in B_R, \\ u(x,t) = 0, \ \ (x,t) \in \vartheta B_R \times (0,T), \\ v \in C(0,T; D(A)), \ \ v_t \in C(0,T; \mathcal{D}^{1,2}), \end{cases}
$$

where by ϑB_R we denote the boundary of the sphere B_R.

Proposition 3.5. Assume that the initial data $u_0 \in D(A)$, $u_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $a = 2$. Then the *linear wave equation* (3.2) has a *unique solution* such that

$$
u \in C\big(0,T; D(A)\big), \ \ u_t \in C\big(0,T; \mathcal{D}^{1,2}(B_R)\big).
$$

Proof. We shall prove existence by means of the classical energy method (*Faedo-Galerkin approximation*). For this we consider the basis of $D(A)$ generated by the eigenfunctions of A (see pp. 48) and we construct an approximating sequence of solutions

$$
u^n(x,t) = \sum_{i=1}^{n} b_{in}(t) w_i,
$$

solving the *Galerkin-system*:

$$
\begin{cases} (u^n_{tt}, w_i)_{L^2_g(B_R)} + \ \|\nabla u^n\|^2 \displaystyle\int_{B_R} \nabla u^n \nabla w_i \, dx + \delta (u^n_t, w_i)_{L^2_g(B_R)} - (|u^n|^2 u^n, w_i)_{L^2_g(B_R)} = 0, \\ u^n(x,0) = \mathcal{P}_n u_0(x), \ \ u^n_t(x,0) = \mathcal{P}_n u_1(x), \end{cases}
$$

where $\mathcal{P}_n$ is the continuous orthogonal projector operator of $D(A) \to span\{w_i : i = 1,2,\dots,n\}$ and of $\mathcal{D}^{1,2}(B_R) \to span\{w_i : i = 1,2,\dots,n\}$[29]. Multiplying the equation by $\dot{b}_{in}(t)$ and summing from 1 to *n*, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|u^n_t\|^2_{L^2_g(B_R)} + \frac{\|\nabla u^n\|^2}{2}\frac{d}{dt}\|u^n\|^2_{\mathcal{D}^{1,2}(B_R)} + \delta \|u^n_t\|^2_{L^2_g(B_R)} = (|u^n|^2 u^n, u^n_t)_{L^2_g(B_R)}.
$$

29. The span of $w_1, w_2, \dots, w_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ is defined to be the set of all *the linear combinations* of these eigenfunctions:

$$
span\{w_1, w_2, \dots, w_n\} \coloneqq \left\{\sum_{i=1}^{n} k_i \, w_i : k_i \in \mathbb{K}, \ \ 1 \le i \le n\right\}
$$

where $\mathbb{K}$ is the scalar-field, which could be the set of the real numbers ($\mathbb{R}$) or of the complex numbers ($\mathbb{C}$).

Since $\|\nabla u^n\|^2 \equiv \|u^n\|^2_{\mathcal{D}^{1,2}(B_R)}$, the equation takes the form

$$\frac{1}{2}\frac{d}{dt}\|u_t^n\|^2_{L_g^2(B_R)} + \frac{1}{4}\frac{d}{dt}\left[\left(\|u^n\|^2_{\mathcal{D}^{1,2}(B_R)}\right)^2\right] + \delta\|u_t^n\|^2_{L_g^2(B_R)} = (|u^n|^2u^n, u_t^n)_{L_g^2(B_R)}.$$

Under the assumption that $|f(u^n)| := ||u^n|^2u^n| \leq c|u^n|^2$, we derive for the last term of the above equality

$$\left|\int_{B_R} \mathrm{g}\, f(u^n)u_t^n dx\right| \leq c\int_{B_R} \mathrm{g}^{1/2}g^{1/2}\,|u^n|^2|u_t^n|dx \;\leq \tilde{c}\int_{B_R} \mathrm{g}\,|u^n|^4dx + \tilde{c}\int_{B_R} g\,|u_t^n|^2dx$$

$$\leq \tilde{c}\left(\int_{B_R} g^a dx\right)^{1/a}\left(\int_{B_R} |\nabla u^n|^2 dx\right)^2 + \tilde{c}\int_{B_R} g\,|u_t^n|^2dx$$

or

$$\left|(|u^n|^2u^n, u_t^n)_{L_g^2(B_R)}\right| \leq \tilde{c}\left\{\|g\|_{L^a(B_R)}\|u^n\|^4_{\mathcal{D}^{1,2}(B_R)} + \|u_t^n\|^2_{L_g^2(B_R)}\right\},$$

where in the second inequality we have used *Cauchy's inequality* (see pp. 9) with $\tilde{c} = c/2$ and in the third the Lemma 3.1. Therefore, according to this we obtain the inequality below where $\mathcal{C} = \mathcal{C}(\tilde{c}, \delta, \|g\|_{L^a(B_R)})$

$$d/dt\left(\|u_t^n\|^2_{L_g^2(B_R)} + \|u^n\|^4_{\mathcal{D}^{1,2}(B_R)}\right) \leq \mathcal{C}\left(\|u^n\|^4_{\mathcal{D}^{1,2}(B_R)} + \|u_t^n\|^2_{L_g^2(B_R)}\right).$$

The rest of the proof follows the steps in (Lemma 3.1, pp. 189-192; [7]). Q.E.D.
Next, we will prove the following theorem.

Theorem 3.6. Assume that $f(u) = |u|^2u$ is a nonlinear C^1*-function* such that $|f'(u)| \leq k_1|u|^2$. If $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ and satisfy the nondegenerate condition

$$\|\nabla u_0\| > 0,$$

then there exists time $T = T\left(\|u_0\|_{D(A)},\ \|\nabla u_1\|^2\right) > 0$, such that the problem $(1.1), (1.2)$ admits a *unique local weak solution* u, satisfying

$$u \in C\big(0, T; D(A)\big),\ \ u_t \in C(0, T; \mathcal{D}^{1,2}).$$

Moreover, at least one of the following statements holds true, either

(i) $T = \infty$, or

(ii) $\lim\limits_{t\to T_-} e(u(t)) \equiv \lim\limits_{t\to T_-}\left(\|u_t(t)\|^2_{\mathcal{D}^{1,2}} + \|u(t)\|^2_{D(A)}\right) = \infty.$

Proof. The proof is based on the *Banach fixed-point theorem* (see pp.24-26).

To apply this theorem, we introduce the two-parameter space of solutions

$$X_{T,R} \coloneqq \left\{ \begin{matrix} v \in C(0,T;D(A)) \colon v_t \in C(0,T;\mathcal{D}^{1,2}), v(0) = u_0, \\ v_t(0) = u_1,\ \ e\big(v(t)\big) \leq R^2,\ \ \forall\ t \in [0,T] \end{matrix} \right\},$$

which is a complete metric space under the distance function

$$d(u,v) \coloneqq \sup_{0 \leq t \leq T} e_1\big(u(t) - v(t)\big),\ \text{where } e_1(v) \coloneqq \|v_t\|^2_{L^2_g} + \|v\|^2_{\mathcal{D}^{1,2}},$$

for any given $T > 0,\ R > 0$.

Next, we introduce the *non-linear mapping* S in the following way. Given $v \in X_{T,R}$ we define $u = Sv$ to be the unique solution of the linear wave equation (3.2). In the following we shall show that there exist $T > 0,\ R > 0$ such that the conditions below to be valid

(a) S maps $X_{T,R}$ into itself, *i.e.*, $S \colon X_{T,R} \to X_{T,R}$.
(b) S is a *contraction* with respect to the metric $d(\cdot\,,\,\cdot)$.

We set $2M_0 \coloneqq \|\nabla u_0\|^2 > 0$ and denote by

$$T_0 \coloneqq \sup\{t \in [0,\infty) \colon \|\nabla v(s)\|^2 > M_0,\ \ for\ 0 \leq s \leq t\ \}.$$

Then we have

$$T_0 > 0 \text{ and } \|\nabla v(t)\|^2 \geq M_0 \text{ for all } t \in [0,T_0]. \tag{3.3}$$

To prove condition (a), we multiply (3.2) by $-2\Delta u_t$ (in the sense of the inner product in the space L^2) and integrate over $\mathbb{R}^N$, to obtain

$$-2\int_{\mathbb{R}^N} \Delta u_t u_{tt}\, dx + 2\|\nabla v\|^2 \int_{\mathbb{R}^N} \varphi(x)\Delta u_t \Delta u\, dx$$
$$- 2\delta \int_{\mathbb{R}^N} \Delta u_t u_t\, dx = -\,2 \int_{\mathbb{R}^N} |v|^2 v \Delta u_t\, dx. \tag{3.4}$$

Setting,

$$e_2^*\big(u(t)\big) \coloneqq \|\nabla u_t(t)\|^2 + \|\nabla v(t)\|^2 \|u(t)\|^2_{D(A)}$$

(3.4) takes the form

$$\frac{d}{dt} e_2^*(u) + 2\delta\|\nabla u_t\|^2 = \left(\frac{d}{dt}\|\nabla v\|^2\right) \|u\|^2_{D(A)} - 2(|v|^2 v, \Delta u_t). \tag{3.5}$$

We observe that the following estimate holds

$$e_2^*(u) \geq \|\nabla u_t(t)\|^2 + M_0 \|u(t)\|^2_{D(A)} \geq c_1^{-2} e(u), \tag{3.6}$$

with $c_1 \coloneqq (\max\{1, M_0^{-1}\})^{1/2}$. To proceed further, we notice that

$$\left(\frac{d}{dt}\|\nabla v\|^2\right)\|u\|_{D(A)}^2 = 2\int_{\mathbb{R}^N} \Delta v v_t \varphi(x) g(x)\,dx \|u\|_{D(A)}^2$$

$$\leq 2\left(\int_{\mathbb{R}^N} \varphi|\Delta v|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^N} g|v_t|^2\,dx\right)^{1/2}\|u\|_{D(A)}^2$$

$$\leq 2\left(\|v\|_{D(A)}^2\right)^{1/2}\left(\|v_t\|_{L_g^2}^2\right)^{1/2}\|u(t)\|_{D(A)}^2$$

$$\leq 2Rk\left(\|v_t\|_{\mathcal{D}^{1,2}}^2\right)e(u(t))$$

$$\leq 2R^2kc_1^2e_2^*(u) \leq c_2R^2e_2^*(u), \tag{3.7}$$

with $c_2 = 2\,kc_1^2$, where k is the constant of the embedding $\mathcal{D}^{1,2} \subset L_g^2$. We also have that

$$-2(|v|^2v, \Delta u_t) = -2\int_{\mathbb{R}^N} |v|^2 v\Delta u_t\,dx = 2\int_{\mathbb{R}^N} \nabla(|v|^2v)\nabla u_t\,dx$$

$$= 2\int_{\mathbb{R}^N} f'(v)\nabla v\nabla u_t\,dx \leq 2k_1\|v\|_{L^{2N}}^2\|\nabla v\|_{L^{\frac{2N}{N-2}}}\|\nabla u_t\|, \tag{3.8}$$

where we used *Hölder inequality* with $p^{-1} = 1/N$, $q^{-1} = (N-2)/2N$ and $r^{-1} = 1/2$. Then, from Lemma 3.1 and the embeddings (2.6) we get

$$\|v\|_{L^{2N}}^2 \leq R^2,\ \ \|\nabla v\|_{L^{\frac{2N}{N-2}}} \leq \|v\|_{D(A)} \leq R,\ \ and\ \ \|\nabla u_t\| \leq e(u)^{1/2}. \tag{3.9}$$

Using estimates (3.7)-(3.9), we get from equation (3.5) that

$$\frac{d}{dt}e_2^*(u) \leq c_2R^2e_2^*(u) + c_3R^3e_2^*\big(u(t)\big)^{1/2},$$

with $c_3 \coloneqq 2k_1c_1$. Hence, from *Gronwall's inequality*, we derive

$$e_2^*(u) \leq e^{\int_0^T c_2R^2\,ds}\left[e_2^*(u(0)) + \int_0^T c_3R^3e_2^*\big(u(s)\big)^{1/2}\,ds\right]$$

$$\leq e^{c_2R^2T}\left[e_2^*\big(u(0)\big)^{1/2} + c_3R^3T\right]^2.$$

According to estimate (3.6), we receive the following relation

$$e(u) \leq c_1^2\left[e_2^*\big(u(0)\big)^{1/2} + c_3R^3T\right]^2 e^{c_2R^2T} \coloneqq \mathfrak{B}_{T,R}^*, \tag{3.10}$$

for any $t \in [0,T]$, with $T \leq T_0$. Therefore, if we assume that

$$\mathfrak{B}^*_{T,R} < R^2,$$

then condition (a) is valid, *i.e.*, S maps $X_{T,R}$ into itself.

To prove condition (b), we take $v_1, v_2 \in X_{T,R}$ and denote by $u_1 = Sv_1$, $u_2 = Sv_2$. Henceforth, we suppose that $\mathfrak{B}^*_{T,R} < R^2$, *i.e.*, $u_1, u_2 \in X_{T,R}$, and set $w = u_1 - u_2$. The function w satisfies the following relation

$$w_{tt} - \varphi\|\nabla v_1\|^2 \Delta w + \delta w_t = \varphi\{\|\nabla v_1\|^2 - \|\nabla v_2\|^2\}\Delta u_2 + |v_1|^2 v_1 - |v_2|^2 v_2,$$

$$w(0) = 0, \qquad w_t(0) = 0.$$

Multiplying this equation by $2gw_t$ and integrating over $\mathbb{R}^N$ we obtain

$$\begin{aligned} 2\int_{\mathbb{R}^N} g w_t w_{tt}\, dx - 2\int_{\mathbb{R}^N} \|\nabla v_1\|^2 \Delta w w_t\, dx + 2\delta \int_{\mathbb{R}^N} g w_t^2\, dx \\ = \ 2\{\|\nabla v_1\|^2 - \|\nabla v_2\|^2\} \int_{\mathbb{R}^N} \Delta u_2 w_t\, dx \\ + 2\int_{\mathbb{R}^N} g\{|v_1|^2 v_1 - |v_2|^2 v_2\} w_t\, dx. \qquad (3.11) \end{aligned}$$

Setting,

$$e^*_{v_1}\big(w(t)\big) \coloneqq \|w_t(t)\|^2_{L^2_g} + \|v_1(t)\|^2_{\mathcal{D}^{1,2}} \|w(t)\|^2_{\mathcal{D}^{1,2}},$$

(3.11) takes the form

$$\begin{aligned} \frac{d}{dt} e^*_{v_1}(w) + 2\delta\|w_t\|^2_{L^2_g} = \frac{d}{dt}(\|\nabla v_1\|^2)\|\nabla w\|^2 + 2\{\|\nabla v_1\|^2 - \|\nabla v_2\|^2\} \\ \times (\Delta u_2, w_t) + 2(|v_1|^2 v_1 - |v_2|^2 v_2, w_t)_{L^2_g} \\ \equiv I_1(t) + I_2(t) + I_3(t). \qquad (3.12) \end{aligned}$$

We also observe that

$$e^*_{v_1}(w) \geq \|w_t\|^2_{L^2_g} + M_0\|w\|^2_{\mathcal{D}^{1,2}} \geq c_1^{-2} e_1(w). \qquad (3.13)$$

As in (3.7), we notice that

$$I_1(t) \leq c_2 R^2 e^*_{v_1}(w) \qquad (3.14)$$

$$I_2(t) \leq 2(R+R)e(v_1 - v_2)^{1/2} \int_{\mathbb{R}^N} |\Delta u_2||w_t|\, dx. \qquad (3.15)$$

For the last term of (3.15), from estimation (3.13), we have that

$$\int_{\mathbb{R}^N} |\Delta u_2||w_t|\varphi^{1/2}\varphi^{1/2} g\, dx \leq \left(\|u_2(t)\|^2_{D(A)}\right)^{1/2} \left(\|w_t(t)\|^2_{L^2_g}\right)^{1/2}$$

$$< Re_1(w(t))^{1/2} < Rc_1 e^*_{v_1}(w)^{1/2}. \qquad (3.16)$$

Therefore, from (3.15) and (3.16), we derive that

$$I_2(t) \leq c_4 R^2\, e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w)^{1/2}, \qquad (3.17)$$

where $c_4 \coloneqq 4c_1$. Applying the *generalized Poincaré's inequality* (Lemma 2.2) and the embeddings (2.6), we obtain

$$I_3(t) \leq 2k_0\xi^{-1}(\|\nabla v_1\|^2 + \|\nabla v_2\|^2)\|\nabla(v_1 - v_2)\|\|w_t\|_{L^2_g}$$

$$\leq c_6 R^2 e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w)^{1/2}, \qquad (3.18)$$

where $c_6 \coloneqq 4k_0\xi^{-1}c_1$, $\xi = k^{-2}\|g\|^{-1}_{N/2}$ the *Poincaré's embedding constant* (see Lemma 2.2) and k_0 is a constant derived from the formula of f. From estimates (3.14), (3.17) and (3.18) we get the following estimate for the relation (3.12)

$$\frac{d}{dt} e^*_{v_1}(w) \leq c_2 R^2 e^*_{v_1}(w) + (c_4 R^2 + c_6 R^2) e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w)^{1/2}.$$

Gronwall's inequality and the fact that $e^*_{v_1}(w(0)) = 0$, imply

$$e^*_{v_1}(w) \leq (c_4 R^2 + c_6 R^2)^2 T^2 e^{c_2 R^2 T} \sup_{0 \leq t \leq T} e_1(v_1(t) - v_2(t)). \qquad (3.19)$$

Therefore, from (3.10) and (3.19), we derive

$$d(u_1, u_2) \leq \mathfrak{B}_{T,R}\, d(v_1, v_2), \qquad (3.20)$$

where

$$\mathfrak{B}_{T,R} \coloneqq 4 \max\left\{1, \frac{\|\nabla u_0\|^{-2}}{2}\right\} R^4 T^2 \left(1 + k_0 k^2 \|g\|_{N/2}\right)^2 e^{2kc_1^2 R^2 T},$$

by substituting c_1, c_2, c_4, c_6. From this we conclude that the map S is a contraction if

$$\mathfrak{B}_{T,R} < 1.$$

We note that the two inequalities $\mathfrak{B}^*_{T,R} < R^2$ and $\mathfrak{B}_{T,R} < 1$, are justified at the same time, if the parameter R is sufficiently large and T is sufficiently small. Applying Banach fixed-point theorem we obtain the local existence result. The second statement of Theorem 3.6 is proved by a standard continuation argument (see pp. 54 in [10]). Q.E.D.

Next, we shall prove that the relation

$$\|\nabla u(t)\| > 0,$$

is valid for all $t \geq 0$. For this we consider the general equation

$$u_{tt} - \varphi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t + f(u) = 0, \;\; x \in \mathbb{R}^N, t \geq 0, \tag{3.21}$$

with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq$ ≥ 3, $\delta \geq 0$ and $f(u)$ a nonlinear C^1-*function* such that

$$\int_{\mathbb{R}^N} f(u)u\, dx \geq k_0^{-1} \int_{\mathbb{R}^N} F(u)\, dx \geq 0, \;\; F(u) \equiv \, 2\int_0^u f(\eta)\, d\eta \tag{3.22}$$

$$|f(u)| \leq k_1 |u|^{a+1}, \;\; |f'(u)| \leq k_2 |u|^a, \tag{3.23}$$

where $k_0, k_1, k_2 \geq 1$ for $a \geq 0$. In our case, where $f(u) = |u|^2 u$, we can take $k_0 = 4$, $k_1 = 1$, and $k_2 = 3$ (see pp. 54 in [10]).

To define the energy related to equation (3.21), we multiply the equation by $2gu_t$ and integrate over $\mathbb{R}^N$ to obtain the following relation (for simplicity we have set $\delta =$ 1)

$$2\int_{\mathbb{R}^N} g u_t u_{tt}\, dx - 2\|\nabla u(t)\|^2 \int_{\mathbb{R}^N} u_t \Delta u dx + 2\int_{\mathbb{R}^N} g u_t^2 dx + 2\int_{\mathbb{R}^N} g u_t |u|^2 u dx = 0.$$

Using some derivative arguments, we derive

$$\frac{d}{dt}\left\{ \|u_t(t)\|_{L_g^2}^2 + \frac{1}{2}\|u(t)\|_{\mathcal{D}^{1,2}}^4 + \int_{\mathbb{R}^N} F(u)\, dx \right\} + 2\|u_t(t)\|_{L_g^2}^2 = 0. \tag{3.24}$$

Then, we define as the energy functional of (3.21) the quantity

$$E(t) \coloneqq E\big(u(t), u_t(t)\big) \coloneqq \|u_t(t)\|_{L_g^2}^2 + \frac{1}{2}\|u(t)\|_{\mathcal{D}^{1,2}}^4 + \int_{\mathbb{R}^N} F(u)\, dx. \tag{3.25}$$

Thereby (3.24) may be written as follows

$$\frac{d}{dt} E(t) + 2\|u_t(t)\|_{L_g^2}^2 = 0. \tag{3.26}$$

From this we observe that the energy functional has a negative derivative which indicates an upper bound. Explicitly, this means that $E(t) \leq E(0)$, since the rate of change is decreasing in time and depends on the initial data. Therefore, we may take the following estimate for the quantity $\|\nabla u(t)\|$:

$$\|\nabla u(t)\| \leq \{2E(t)\}^{1/4} \leq \{2E(0)\}^{1/4}. \tag{3.27}$$

Having made these notations, we are able to prove the following lemma.

Lemma 3.7. Assume that $f(u) = |u|^2 u$ is a nonlinear C^1-*function*, and $N \geq 3$. If the initial data $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ and satisfy the condition

$$\|\nabla u_0\| \neq 0,$$

then we have that

$$\|\nabla u(t)\| > 0, \quad \forall\, t \geq 0.$$

Proof. Consider $u(t)$ the unique solution of (3.21), by Theorem 3.6 in the space $[0, T)$. Multiplying equation (3.21) by $-2\Delta u_t$ and integrating over $\mathbb{R}^N$ we derive

$$-2\int_{\mathbb{R}^N} \Delta u_t u_{tt}\, dx + 2\|\nabla u(t)\|^2 \int_{\mathbb{R}^N} \varphi(x)\Delta u_t \Delta u\, dx$$
$$-2\delta \int_{\mathbb{R}^N} u_t \Delta u_t dx - 2\int_{\mathbb{R}^N} |u|^2 u \Delta u_t\, dx = 0.$$

Setting for simplicity $\delta = 1$ and using some derivative arguments we obtain the following relation

$$\frac{d}{dt}\|\nabla u_t(t)\|^2 + \|\nabla u(t)\|^2 \frac{d}{dt}\|u(t)\|^2_{D(A)}$$
$$+ 2\|\nabla u_t(t)\|^2 + 2(|u|^2 u, \Delta u_t(t)) = 0. \tag{3.28}$$

Since $\|\nabla u_0\| > 0$ and $\|\nabla u_0\| \neq 0$, we observe that $\|\nabla u(t)\| > 0$ near $t = 0$. Let

$$T \coloneqq \sup\{t \in [0, \infty) \colon \|\nabla u(s)\| > 0, \;\; for\ 0 \leq s < t\,\}.$$

Thus $T > 0$ and $\|\nabla u(t)\| > 0$ for $0 \leq t < T$. If we assume that $T < \infty$, then we have

$$\lim_{t \to T_-} \|\nabla u(t)\| = 0. \tag{3.29}$$

Making a variable change $(t \coloneqq T - t \equiv \tau)$, we derive that the function $\tilde{u}(t) = u(T - t) = u(\tau)$, satisfies the problem:

$$\tilde{u}_{tt}(t) - \varphi(x)\|\nabla\tilde{u}(t)\|^2 \Delta\tilde{u}(t) - \tilde{u}_t(t) + f\big(\tilde{u}(t)\big) = 0, \;\; x \in \mathbb{R}^N, t \geq 0, \tag{3.30}$$

$$\tilde{u}(0) = 0 \;\text{ and } \tilde{u}_t(0) = 0, \;\; x \in \mathbb{R}^N,$$

where $u' = du/d\tau$.
We observe that $\tilde{u} \in C^0([0, T]; D(A)) \cap C^1([0, T]; \mathcal{D}^{1,2})$. Multiplying equation (3.30) by $2g\tilde{u}_t$ and integrating over $\mathbb{R}^N$, we obtain an equation analogous to (3.26):

$$\frac{d}{dt}E\big(\tilde{u}(t),\tilde{u}_t(t)\big) = 2\|u_t(t)\|^2_{L^2_g} \leq 2E\big(\tilde{u}(t),\tilde{u}_t(t)\big). \tag{3.31}$$

Integrating (3.31) over $[0,t]$, we obtain that

$$E\big(\tilde{u}(t),\tilde{u}_t(t)\big) \leq 2\int_0^t E\big(\tilde{u}(s),\tilde{u}'(s)\big)\,ds,$$

for $0 \leq t \leq T$. Since $E\big(\tilde{u}(0),\tilde{u}'(0)\big) = 0$, we can apply Gronwall's inequality to derive that

$$E\big(\tilde{u}(t),\tilde{u}'(t)\big) = 0, \ \ \forall\, t \in [0,T],$$

i.e., $\|\nabla u(T-t)\| = 0$ at $[0,T]$ which comes in contradiction with $\|\nabla u_0\| \neq 0$. Therefore, $T = \infty$ and $\|\nabla u(t)\| > 0$, for all $t \geq 0$. Q.E.D.

4. Global solution and energy estimates

Two subjects related with the asymptotic behavior of the solutions of evolutionary equations are the existence of global solutions and the blow-up phenomena. In this section we study the global solution and energy estimate for the initial value problem $(1.1), (1.2)$ and in the next section the existence of blow-up phenomena.

Let us consider for the moment that there exists a maximally defined with respect to time function/solution of $(1.1), (1.2)$ with the associated mapping $\boldsymbol{u}: [0,T] \to D(A)$ defining by

$$[\boldsymbol{u}(t)](x) \coloneqq u(x,t) \ \ (x \in \mathbb{R}^N, 0 \leq t \leq T).$$

If $T < \infty$, then the solution is referred as *local* as we have already seen in preceding sections, but if we can determine the exact conditions which must be satisfied by the initial data of our problem and the functional $E(\cdot)$ -which is related with the energy function of the physical phenomenon behind the mathematical prescription-, such that the local solution to be expanded in $[0,\infty)$, then we have answered the questions concerning global existence issues.

The method that we will follow in order to prove the global existence is the *potential well method*[30] which may be understood in the following sense:

"Consider a particle moving in $\mathbb{R}$ in the presence of a potential V that is zero for $0 \leq x \leq L$ and Q (which denotes a sufficiently large constant value) on the rest of the real line. According to classical mechanics the motion in the area of Q is valid *if and only if* $\mathcal{E} > Q$, where $\mathcal{E}$ denotes the energy of the particle. In quantum mechanics the result

30. See more in the paper "*Saddle Points and Instability of Non-linear Hyperbolic Equations*" by L. E. Payne and D. H. Sattinger; Israel J math., 22, (1975), 273-303.

is similar and the validity of motion outside of $0 \leq x \leq L$ is interpreted by means of wave functions, which correspond to each point of the real line intensity probabilities. More explicitly if we have a solution of the time-independent Schrödinger equation[31] $\widehat{\mathcal{H}}\psi = \mathcal{E}\psi$ for this potential (with $\mathcal{E} \ll \mathcal{Q}$), then we expect the wave function to decay rapidly for x outside of the "box". Especially in the case where $\mathcal{Q} \to \infty$, $\psi \equiv 0$ for $x \in (-\infty, 0) \cup (L, \infty)$ and $\psi \to 0$ as $x \to 0, L$ (see Figure 3).

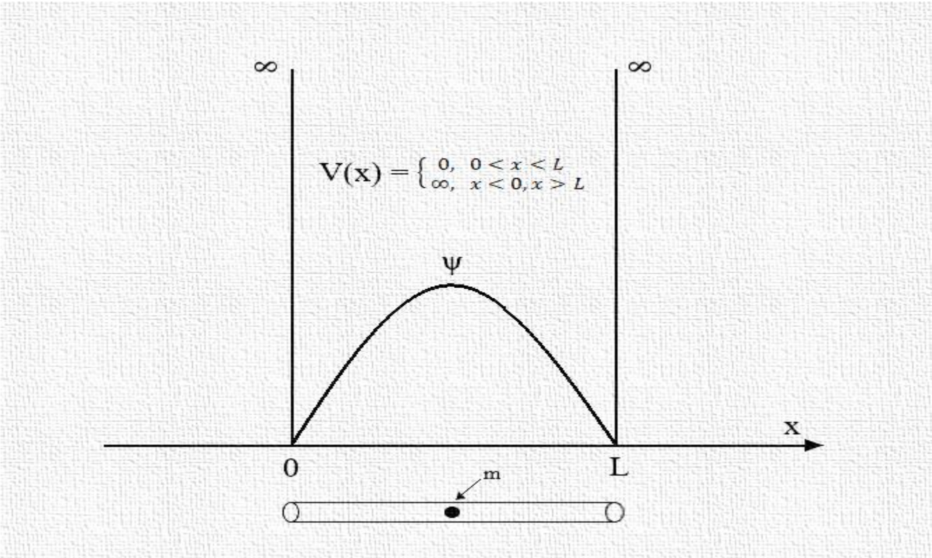

Figure 3: Infinite potential well

The above prescription relates the behavior of a particle moving in $\mathbb{R}$ under the predefined potential with the existence of global solution, in the sense of the enclosure of the solution in a potential well, in order to obtain the valid solvability expansion. Obviously, if $\mathcal{E} > \mathcal{Q}$ the particle can move in the region outside of the "box" and in that case the stability and global existence questions are open".

From this correlation we observe that the determination of the energy and potential for the problem (1.1), (1.2) is required to proceed with the analysis of the global existence. For this we multiply equation (1.1) by $2gu_t$ and integrate over $\mathbb{R}^N$ to obtain the following equation

$$\frac{d}{dt}\left\{\|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{1}{2}\|u(t)\|^4_{L^4_g}\right\} + 2\delta\|u_t(t)\|^2_{L^2_g} = 0. \qquad (4.1)$$

Therefore, we define as the energy of the problem (1.1), (1.2) the quantity:

31. See Chapter 3 in [1].

$$E(t) \coloneqq E\big(u(t), u_t(t)\big) \coloneqq \|u_t(t)\|_{L_g^2}^2 + \frac{1}{2}\|u(t)\|_{\mathcal{D}^{1,2}}^4 - \frac{1}{2}\|u(t)\|_{L_g^4}^4. \tag{4.2}$$

Hence, equation (4.1) may be written as

$$\frac{d}{dt}E(t) + 2\delta\|u_t(t)\|_{L_g^2}^2 = 0. \tag{4.3}$$

Furthermore, we introduce the potential of the problem (1.1), (1.2) as follows

$$\mathcal{P}(u) \coloneqq \frac{1}{2}\|u(t)\|_{\mathcal{D}^{1,2}}^4 - \frac{1}{2}\|u(t)\|_{L_g^4}^4. \tag{4.4}$$

So, from equation (4.1) and definitions (4.2), (4.4) we derive the relation below

$$E(t) = \|u_t(t)\|_{L_g^2}^2 + \mathcal{P}(u). \tag{4.5}$$

Finally, we introduce a version of the *modified potential well*[32], by

$$\mathcal{W} \coloneqq \Big\{u \in D(A);\ \mathcal{K}(u) \coloneqq \|u(t)\|_{\mathcal{D}^{1,2}}^4 - \|u(t)\|_{L_g^{a+2}}^{a+2} > 0\Big\} \cup \{0\}. \tag{4.6}$$

Thereinafter we give two auxiliary lemmas concerning the behavior of the potential well.

Lemma 4.1. If $2 < a < 4/(N-2)$, then $\mathcal{W}$ is an open neighborhood of 0 in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. Since $2 < a < 4/(N-2)$, by Lemma 3.3 and Lemma 2.2 we have that

$$\begin{aligned}
\|u\|_{L_g^{a+2}}^{a+2} &\le C_0\|u\|_{L_g^2}^{(1-\theta)(a+2)}\|u\|_{\mathcal{D}^{1,2}}^{(a+2)\theta} \\
&\le C_0\|u\|_{L_g^2}^{(1-\theta)(a+2)}\|u\|_{\mathcal{D}^{1,2}}^{(a+2)\theta-4}\|u\|_{\mathcal{D}^{1,2}}^4 \\
&\le C_0\xi^{-1}\|u\|_{\mathcal{D}^{1,2}}^{(1-\theta)(a+2)}\|u\|_{\mathcal{D}^{1,2}}^{(a+2)\theta-4}\|u\|_{\mathcal{D}^{1,2}}^4 \\
&\le C_0\xi^{-1}\|u\|_{\mathcal{D}^{1,2}}^{a-2}\|u\|_{\mathcal{D}^{1,2}}^4.
\end{aligned} \tag{4.7}$$

Therefore, from (4.7) we derive

$$\mathcal{K}(u) \coloneqq \|u\|_{\mathcal{D}^{1,2}}^4 - \|u\|_{L_g^{a+2}}^{a+2} \ge \big(1 - C_0\xi^{-1}\|u\|_{\mathcal{D}^{1,2}}^{a-2}\big)\|u\|_{\mathcal{D}^{1,2}}^4. \tag{4.8}$$

Hence, if the following is valid

$$C_0\xi^{-1}\|u\|_{\mathcal{D}^{1,2}}^{a-2} \le 1 \Longrightarrow \|u\|_{\mathcal{D}^{1,2}}^{a-2} \le \xi C_0^{-1} \Longrightarrow \|u\|_{\mathcal{D}^{1,2}} \le (\xi C_0^{-1})^{\frac{1}{a-2}},$$

where by substituting $\xi = k^{-2}\|g\|_{N/2}^{-1}$ and $C_0 = k^{\theta}$ we obtain

32. See pp. 40-41, and 57 in [10].

$$\|u\|_{\mathcal{D}^{1,2}} \leq \left(k^{-2-\theta}\|g\|_{N/2}^{-1}\right)^{\frac{1}{a-2}},$$

then $\mathcal{K}(u) \geq 0$ and 0 belongs in $\mathcal{W}$. Q.E.D.

Remark 4.2. The condition $2 < a < 4/(N-2)$ implies that $N = 3$.

Remark 4.3. In the limit case where $a = 2$ we observe that, $\|u\|_{L_g^4}^4 \leq C_0\xi^{-1}\|u\|_{\mathcal{D}^{1,2}}^4$ and $\mathcal{K}(u) \geq (1 - C_0\xi^{-1})\|u\|_{\mathcal{D}^{1,2}}^4$. This means that we have a degeneration as $\mathcal{K}(u) \geq 0$ and 0 belongs in $\mathcal{W}$ if $k^{\theta+2}\|g\|_{N/2} \leq 1$ is valid for $\theta \in (0,1)$.

Lemma 4.4. If $u \in \mathcal{W}$, $N = 3$ and $a > 2$, then we have

$$0 \leq \frac{a-2}{2(a+2)}\|u\|_{\mathcal{D}^{1,2}}^4 \leq \tilde{\mathcal{P}}(u) \leq \tilde{E}(u, u_t), \tag{4.9}$$

where $\tilde{\mathcal{P}}(u) \coloneqq \frac{1}{2}\|u(t)\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}\|u(t)\|_{L_g^{a+2}}^{a+2}$ and $\tilde{E}(u, u_t) \coloneqq \|u_t(t)\|_{L_g^2}^2 + \tilde{\mathcal{P}}(u)$, the potential and energy of the problem (1.1), (1.2) with $f(u) = |u|^a u$, respectively.

Proof. Since $a > 2$, from the definitions of the potential and of the modified potential well, for any $u \in \mathcal{W}$, we have that

$$\begin{aligned}
\tilde{\mathcal{P}}(u) &= \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}\|u\|_{L_g^{a+2}}^{a+2} \\
&\geq \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}C_0\xi^{-1}\|u\|_{\mathcal{D}^{1,2}}^{a-2}\|u\|_{\mathcal{D}^{1,2}}^4 \\
&\geq \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}C_0\xi^{-1}\xi C_0^{-1}\|u\|_{\mathcal{D}^{1,2}}^4 \\
&\geq \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}\|u\|_{\mathcal{D}^{1,2}}^4 \equiv \frac{\alpha-2}{2(\alpha+2)}\|u\|_{\mathcal{D}^{1,2}}^4.
\end{aligned}$$

Therefore, $\tilde{\mathcal{P}}(u) \geq \frac{\alpha-2}{2(\alpha+2)}\|u\|_{\mathcal{D}^{1,2}}^4$. Q.E.D.

Remark 4.5. In the limit case where $a = 2$ we observe that

$$\begin{aligned}
\mathcal{P}(u) &= \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^4 - \frac{1}{2}\|u\|_{L_g^4}^4 \\
&\geq \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^4 - \frac{1}{2}C_0\xi^{-1}\|u\|_{\mathcal{D}^{1,2}}^4 \\
&\geq \left(\frac{1 - k^{\theta+2}\|g\|_{N/2}}{2}\right)\|u\|_{\mathcal{D}^{1,2}}^4
\end{aligned}$$

and that the relation (4.9) is reduced to $0 \leq m\|u\|^4_{\mathcal{D}^{1,2}} \leq \mathcal{P}(u) \leq E(t)$, where $m :=$ $\left(1 - k^{\theta+2}\|g\|_{N/2}\right)/2$, for $\theta \in (0,1)$.

Concerning the time behavior of the energy we have the following remarks.
Integrate equation (4.3) over $[0,t]$, to derive

$$E(t) + 2\delta \int_0^t \|u_t(t)\|^2_{L^2_g}\, dx = E(0). \tag{4.10}$$

Let us note that, if $u \in \mathcal{W}$, then by definition $E(u,u_t) \geq 0$, whereas, if $E(u,u_t) < 0$, then $u \notin \overline{\mathcal{W}}$. From equation (4.3) and definition (4.2) we obtain that

$$\frac{d}{dt}E(u,u_t) = -2\delta\|u_t(t)\|^2_{L^2_g} \leq 0. \tag{4.11}$$

Therefore, the energy $E(t)$ is a nonincreasing function of t. Hence, we have that

$$E(t) \leq E(0), \ \ \forall\, t \in [0,T). \tag{4.12}$$

In the theorem below we shall prove the global existence and the energy decay properties for the problem (1.1), (1.2).

Theorem 4.6. Assume that $N = 3$, $8/3 < a < 4$, $u_0 \in \mathcal{W}(\subset D(A))$ and $u_1 \in \mathcal{D}^{1,2}$. Also suppose that the following inequality is valid

$$E(0) \leq \left(\frac{1}{C_0\mu_0^{p_1}}\right)^{1/p_2} \ \ if \ \ 8/3 < a < 4 \ and \ p_2 > 0. \tag{4.13}$$

Then

a) for $p_1 := (2(a+2) - 3a)/2$ and $p_2 := (3a-8)/8$ there *exists a unique global solution* $u \in \mathcal{W}$ of the problem (1.1), (1.2) satisfying

$$u \in C\big([0,\infty); D(A)\big) \ \ and \ \ u_t \in C([0,\infty);\ \mathcal{D}^{1,2}(\mathbb{R}^N)). \tag{4.14}$$

b) Moreover, this solution satisfies the following estimate

$$\|u_t\|^2_{L^2_g} + d_*^{-1}\|\nabla u\|^4 \leq E(u,u_t) \leq \left\{E(u_0,u_1)^{-1/2} + d_0^{-1}[t-1]^+\right\}^{-2}, \tag{4.15}$$

where $d_* := 2(a+2)/(a-2)$ and $d_0 \geq 1$, *i.e.*,

$$\|\nabla u\|^4 \leq C_*(1+t)^{-1}, \tag{4.16}$$

where C_* is some constant depending on $\|u_0\|^4_{\mathcal{D}^{1,2}}$ and $\|u_1\|_{L^2_g}$.

Proof. (a) To show that the local solution given by Theorem 3.6, remains in the modified potential well $\mathcal{W}$, as long as it exists, we shall argue by contradiction.

Assume that there exists time $T^* > 0$, such that $u(t) \in \mathcal{W}$, where $0 \leq t < T^*$ and $u(T^*) \in \vartheta\mathcal{W}$. Then $\mathcal{K}(u(T^*)) = 0$ and $u(T^*) \neq 0$. We multiply equation (1.1) by gu and integrate over $\mathbb{R}^N$ to obtain the equation

$$\int_{\mathbb{R}^N} guu_{tt}\,dx - \|\nabla u(t)\|^2 \int_{\mathbb{R}^N} u\Delta u\,dx + \delta \int_{\mathbb{R}^N} guu_t\,dx = \int_{\mathbb{R}^N} g|u|^a u^2\,dx.$$

Using some derivative arguments, this gives

$$\frac{d}{dt}\big(u(t),u_t(t)\big)_{L_g^2} - \|u_t(t)\|_{L_g^2}^2 + \frac{\delta}{2}\frac{d}{dt}\|u(t)\|_{L_g^2}^2 + \|u(t)\|_{\mathcal{D}^{1,2}}^4 - \int_{\mathbb{R}^N} g(x)|u(t)|^{a+2}\,dx = 0. \tag{4.17}$$

By integrating (4.17) over $[0,t]$, for some $t \in [0,T)$, we derive

$$\int_0^t \left\{\frac{d}{dt}\big(u(s),u_t(s)\big)_{L_g^2}\right\} ds - \int_0^t \|u_t(s)\|_{L_g^2}^2\,ds + \frac{\delta}{2}\int_0^t \left\{\frac{d}{dt}\|u(s)\|_{L_g^2}^2\right\} ds + \int_0^t \|u(s)\|_{\mathcal{D}^{1,2}}^4\,ds - \int_0^t \left\{\int_{\mathbb{R}^N} g(x)|u(s)|^{a+2}\,dx\right\} ds = 0,$$

or

$$\big(u(t),u_t(t)\big)_{L_g^2} - \big(u(0),u_t(0)\big)_{L_g^2} - \int_0^t \|u_t(s)\|_{L_g^2}^2\,ds + \frac{\delta}{2}\left\{\|u(t)\|_{L_g^2}^2 - \|u(0)\|_{L_g^2}^2\right\} + \int_0^t \|u(s)\|_{\mathcal{D}^{1,2}}^4\,ds - \int_0^t \left\{\int_{\mathbb{R}^N} g(x)|u(s)|^{a+2}\,dx\right\} ds = 0.$$

Using Young's inequality for $\varepsilon = \delta/2$ in the first term of the last relation we get

$$\delta\|u(t)\|_{L_g^2}^2 \leq \delta\|u(0)\|_{L_g^2}^2 + 2\left(\frac{\delta}{4}\|u(t)\|_{L_g^2}^2 + \frac{1}{\delta}\|u_t(t)\|_{L_g^2}^2\right) + 2(u_0,u_1)_{L_g^2} + 2\int_0^t \|u_t(s)\|_{L_g^2}^2\,ds. \tag{4.18}$$

Since $u(t) \in \mathcal{W}$, we integrate equation (4.1) taking a in the general case, *i.e.*,

$$\frac{d}{dt}\left\{\|u_t(t)\|_{L_g^2}^2 + \frac{1}{2}\|u(t)\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}\|u(t)\|_{L_g^{a+2}}^{a+2}\right\} + 2\delta\|u_t(t)\|_{L_g^2}^2 = 0,$$

over $[0,t]$ to receive:

$$\|u_t(t)\|_{L_g^2}^2 - \|u_1\|_{L_g^2}^2 + \frac{1}{2}\|u(t)\|_{\mathcal{D}^{1,2}}^4 - \frac{1}{2}\|u_0\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}\|u(t)\|_{L_g^{a+2}}^{a+2}$$
$$+\frac{2}{a+2}\|u_0\|_{L_g^{a+2}}^{a+2} + 2\delta\int_0^t \|u_t(s)\|_{L_g^2}^2\, ds = 0.$$

From definition (4.2), where we have used for simplicity the same notation for the energy in the general case of a and in the discrete case ($a = 2$), we have that

$$E(0) \coloneqq E\big(u(0), u_t(0)\big) \coloneqq \|u_1\|_{L_g^2}^2 + \frac{1}{2}\|u_0\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}\|u_0\|_{L_g^{a+2}}^{a+2}.$$

Therefore, from the previous relation we obtain the following estimate

$$\frac{1}{2}\|u_t(t)\|_{L_g^2}^2 + \delta\int_0^t \|u_t(s)\|_{L_g^2}^2\, ds \leq E(0). \tag{4.19}$$

From relations (4.18), (4.19) we get that

$$\|u(t)\|_{L_g^2}^2 \leq \frac{2}{\delta}\left\{\delta\|u(0)\|_{L_g^2}^2 + 2(u_0, u_1)_{L_g^2} + \frac{4}{\delta}E(0)\right\} \coloneqq \mu_0^2. \tag{4.20}$$

Using Lemma 3.3 and relation (4.20) we obtain the inequality

$$\begin{aligned}
\|u(t)\|_{L_g^{a+2}}^{a+2} &\leq C_0\mu_0^{(a+2)(1-\theta)}\|u(t)\|_{\mathcal{D}^{1,2}}^{(a+2)\theta} \\
&\leq C_0\mu_0^{(a+2)(1-\theta)}\|u(t)\|_{\mathcal{D}^{1,2}}^{(a+2)\theta-4}\|u(t)\|_{\mathcal{D}^{1,2}}^4 \\
&\leq C_0\mu_0^{(a+2)(1-\theta)}\tilde{\mathcal{P}}(u)^{\frac{(a+2)\theta}{4}-1}\|u(t)\|_{\mathcal{D}^{1,2}}^4 \\
&\leq C_0\mu_0^{(a+2)(1-\theta)}\tilde{E}(0)^{\frac{(a+2)\theta}{4}-1}\|u(t)\|_{\mathcal{D}^{1,2}}^4,
\end{aligned} \tag{4.21}$$

where, according to Lemma 3.3, the constants are

$$\begin{cases}
\theta \coloneqq \dfrac{3a}{2(a+2)}, \\
p_1 \coloneqq (a+2)(1-\theta) = \dfrac{2(a+2)-3a}{2}, \\
p_2 \coloneqq \dfrac{(a+2)\theta}{4} - 1 = \dfrac{3a-8}{8}.
\end{cases}$$

Thus, we have that

$$\|u(t)\|_{L_g^{a+2}}^{a+2} \leq C_0\mu_0^{p_1}\tilde{E}(0)^{p_2}\|u(t)\|_{\mathcal{D}^{1,2}}^4. \tag{4.22}$$

Assume that the hypothesis (4.13) is valid. Then we get that, $C_0\mu_0^{p_1}\tilde{E}(0)^{p_2} \leq 1$. Setting $\delta_1 \coloneqq C_0\mu_0^{p_1}\tilde{E}(0)^{p_2}$, for $t = T^*$, the inequality (4.21) implies

$$\mathcal{K}\big(u(T^*)\big) = \|u(T^*)\|_{\mathcal{D}^{1,2}}^4 - \|u(T^*)\|_{L_g^{a+2}}^{a+2}$$

$$\geq \|u(T^*)\|_{\mathcal{D}^{1,2}}^4 - \delta_1\|u(T^*)\|_{\mathcal{D}^{1,2}}^4 = (1-\delta_1)\|u(T^*)\|_{\mathcal{D}^{1,2}}^4 > 0, \qquad (4.23)$$

which contradicts the preceding assumption that, $\mathcal{K}(u(T^*)) = 0$.
(b) To show the decay condition of the energy $E(t)$ associated with equation (1.1), we assume for simplicity that $\delta = 1$. Integrating equation (4.3) over $[t, t+1]$, we obtain

$$2\int_t^{t+1} \|u_t(s)\|_{L_g^2}^2\, ds = E(t) - E(t+1)\ \big(\coloneqq 2D^2(t)\big). \qquad (4.24)$$

Therefore, there exist $t_1 \in [t, t+1/4]$, $t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t_i)\|_{L_g^2} \leq 2D(t) \ \text{ for } i = 1,2. \qquad (4.25)$$

Multiplying equation (1.1) by gu and integrating over $\mathbb{R}^N$, we have that

$$\int_{\mathbb{R}^N} g u u_{tt}\, dx - \|\nabla u(t)\|^2 \int_{\mathbb{R}^N} u\Delta u\, dx + \delta \int_{\mathbb{R}^N} g u u_t\, dx = \int_{\mathbb{R}^N} g|u|^a u^2\, dx.$$

For simplicity setting $\delta = 1$ and using some derivative arguments we obtain

$$\frac{d}{dt}\big(u(t), u_t(t)\big)_{L_g^2} - \|u_t(t)\|_{L_g^2}^2 + \|u(t)\|_{\mathcal{D}^{1,2}}^4 + \big(u(t), u_t(t)\big)_{L_g^2} = \|u(t)\|_{L_g^{a+2}}^{a+2}.$$

From (4.6) the above relation takes the form

$$\mathcal{K} = \|u_t(t)\|_{L_g^2}^2 - \frac{d}{dt}\big(u(t), u_t(t)\big)_{L_g^2} - \big(u(t), u_t(t)\big)_{L_g^2}. \qquad (4.26)$$

Integrating (4.26) over $[t_1, t_2]$, it follows from (4.23), (4.24) and (4.25) that

$$\frac{1}{2}\int_{t_1}^{t_2} \|u(s)\|_{\mathcal{D}^{1,2}}^4\, ds \leq \int_{t_1}^{t_2} \mathcal{K}\big(u(s)\big) ds \leq \int_t^{t+1} \|u_t(s)\|_{L_g^2}^2\, ds$$

$$+\left\{\left(\int_t^{t+1} \|u_t(s)\|_{L_g^2}^2\, ds\right)^{\frac{1}{2}} + \sum_{i=1}^{2} \|u_t(t_i)\|_{L_g^2}\right\} \sup_{t\leq s\leq t+1} \|u(s)\|_{L_g^2}$$

$$\leq D^2(t) + 5D(t)a^{-1}(d_* E(t))^{1/4}, \qquad (4.27)$$

where $d_* \coloneqq 2(a+2)/(a-2)$ and the Lemma 4.4 is used in the last inequality.
Therefore, from (4.5), (4.24) and (4.27) we have that

$$\int_{t_1}^{t_2} E(s) ds \leq \int_{t_1}^{t_2} \left\{\|u_t(s)\|_{L_g^2}^2 + \|u(s)\|_{\mathcal{D}^{1,2}}^4\right\} ds \leq D^2(t) + 2(D^2(t) + 5D(t)a^{-1}(d_* E(t))^{1/4}),$$

or

$$\int_{t_1}^{t_2} E(s)ds \leq 3D^2(t) + 10D(t)a^{-1}(d_* E(t))^{1/4}. \tag{4.28}$$

On the other hand, integrating (4.3) -for $\delta = 1$- over $[t, t_2]$ and using (4.24), (4.28) we obtain that

$$\begin{aligned} E(t) &= E(t_2) + 2\int_t^{t_2} \|u_t(s)\|_{L_g^2}^2 \, ds \\ &\leq 2\int_{t_1}^{t_2} E(s)\, ds + 2\int_t^{t_2} \|u_t(s)\|_{L_g^2}^2 \, ds \\ &\leq 2(3D^2(t) + 10D(t)a^{-1}(d_* E(t))^{1/4}) + \ 2D^2(t) \\ &\leq 8D^2(t) + \frac{\varepsilon_1}{2} d_*^2 (20a^{-1}D(t))^{4/3} + (2\varepsilon_1)^{-1}E(t), \end{aligned}$$

where Young's inequality is used for $p^{-1} = 3/4$ and $q^{-1} = 1/4$. Hence

$$E(t) \leq 2\left\{8D^{2/3}(t) + d_*^2(20a^{-1})^{4/3}\right\}D^{4/3}(t). \tag{4.29}$$

Since $2D^2(t) = E(t) - E(t+1) \leq E(t) \leq E(0)$ (≤ 1), it follows from (4.29) that

$$E(t) \leq 2\left\{8(E(0)/2)^{1/3} + d_*^2(20a^{-1})^{4/3}\right\}D^{4/3}(t) = C_5\, D^{4/3}(t), \tag{4.30}$$

where $C_5 \coloneqq 2\left\{8(E(0)/2)^{1/3} + d_*^2(20a^{-1})^{4/3}\right\}$. Also, from relation (4.24) we have that

$$D^{4/3}(t) = 2^{-2/3}(E(t) - E(t+1))^{2/3}. \tag{4.31}$$

Thus from (4.31), relation (4.30) becomes

$$E^{3/2}(t) \leq 2^{-1}C_5^{3/2}\big(E(t) - E(t+1)\big). \tag{4.32}$$

To complete our proof, we shall use the following Lemma (see pp. 61 in [10]).

Lemma 4.7. Let φ be a *non-increasing* and *non-negative* function on $[0, \infty)$ satisfying

$$\sup_{t \leq s \leq t+1} \varphi(s)^{1+r} \leq k\{\varphi(t) - \varphi(t+1)\},$$

for $r > 0$ and $k > 0$. Then

$$\varphi(t) \leq \{\varphi(0)^{-r} + rk^{-1}[t-1]^+\}^{-1/r},$$

for $r \geq 0$. ∎

Therefore, applying Lemma 4.7 we can derive the decay estimate of the energy $E(t)$, such that

$$E(t) \leq \left\{E(0)^{-1/2} + d_0^{-1}[t-1]^{+}\right\}^{-2}, \tag{4.33}$$

where $d_0 \coloneqq 2^{3/2}\left\{8(E(0)/2)^{1/3} + d_*^2(20a^{-1})^{4/3}\right\}^{3/2}$ (≥ 1). Hence,

$$\|\nabla u\|^4 \leq C_*(1+t)^{-1},$$

with some constant C_* depending on $\|u_0\|_{\mathcal{D}^{1,2}}^4$ and $\|u_1\|_{L_g^2}$. Q.E.D.

Remark 4.8. In the case where $a = 2$, we have the following observations:

(i) To derive (4.18) we use the estimate

$$-\int_{\mathbb{R}^N} g(x)|u(t)|^4\,dx \leq -\left(\int_{\mathbb{R}^N} g^\alpha\,dx\right)^{\frac{1}{\alpha}}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2$$
$$= -\left(\int_{\mathbb{R}^N} g^\alpha\,dx\right)^{\frac{1}{\alpha}}\|\nabla u\|^2\int_{\mathbb{R}^N}|\nabla u|^2\,dx,$$

where $\alpha = 2N/(8-2N)$ from Lemma 3.1 for $p = 4$ and the assumption that the equality is valid for $\mathcal{r} = 1$, setting

$$\mathcal{r} \coloneqq \left(\int_{\mathbb{R}^N} g^\alpha\,dx\right)^{1/\alpha},$$

to obtain the identity

$$\|u(t)\|_{\mathcal{D}^{1,2}}^4 - \int_{\mathbb{R}^N} g(x)|u(t)|^4 dx = 0,$$

in relation (4.17).

(ii) In relation (4.21), the constants take the values,

$$\begin{cases} \theta = 3/4\,, \\ p_1 = 1, \\ p_2 = -1/4\,. \end{cases}$$

Hence, (4.22) takes the following form

$$\|u(t)\|_{L_g^4}^4 \leq C_0\mu_0 E(0)^{-1/4}\|u(t)\|_{\mathcal{D}^{1,2}}^4,$$

whereas we notice that the estimate (4.13) is modified, since $p_2 < 0$.

(iii) According to Remark 4.5 the last inequality in relation (4.27) becomes

$$
\begin{aligned}
\frac{1}{2}\int_{t_1}^{t_2}\|u(s)\|^4_{\mathcal{D}^{1,2}}\,ds &\le \int_{t_1}^{t_2}\mathcal{K}\big(u(s)\big)ds \\
&\le \int_t^{t+1}\|u_t(s)\|^2_{L^2_g}\,ds + \left\{\left(\int_t^{t+1}\|u_t(s)\|^2_{L^2_g}\,ds\right)^{\frac{1}{2}}\right. \\
&\qquad \left. + \sum_{i=1}^{2}\|u_t(t_i)\|_{L^2_g}\right\}\sup_{t\le s\le t+1}\|u(s)\|_{L^2_g} \\
&\le D^2(t) + 5D(t)\xi^{-1}\|u\|_{\mathcal{D}^{1,2}} \le\ D^2(t) + 5D(t)\xi^{-1}\big(m^{-1}E(u)\big)^{\frac{1}{4}},
\end{aligned}
$$

where the constants[33] $\xi(\triangleq a)$ and m, are the fixed constants in Lemma 2.2 and Remark 4.5 respectively, *i.e.*, $\xi := k^{-2}\|g\|^{-1}_{N/2}$ and $m := \big(1 - k^{\theta+2}\|g\|_{N/2}\big)/2$. From this we obtain the rest of the proof with the difference that d_* is replaced by m and the notation ξ by a.

5. Blow-up results

In this section we complete our study with the blowing-up property of the solution for the initial value problem (1.1), (1.2). As in the preceding section where we adapt the method of the *modified potential well* to show the global-existence of our solution in the same way we adopt the *concavity method* introduced by Levine in [3], to study the blow-up properties of the solution.
The method is based on the following Theorem. For the proof see Theorem I in [3].

Theorem 5.1. Consider the initial value problem

$$
\begin{cases} \mathcal{P}u_{tt} = -\mathcal{A}(t)u(t) + \mathcal{F}\big(u(t)\big), & t \in [0,T) \\ u(0) = u_0, \quad u_t(0) = v_0, \end{cases}
$$

with the following hypotheses:
(A-I) $\mathcal{A}(t): D \subseteq H \to H$, is a symmetric linear operator, H a Hilbert space with the associated inner product $(\ ,\ )$ and norm $\|\ \|$, respectively, and D a dense Hilbert subspace with respect to $(\ ,\ )_D$, continuously embedded in H.
(A-II) $(x, \mathcal{A}(t)x) \ge 0$ if $x \in D$.
(A-III) If $v: [0,\infty) \to H$ is strongly continuously differentiable and if for all $t \ge 0$, $v(t)$ and $dv(t)/dt \in D$, then $(v(t), \mathcal{A}(t)v(t))$ is continuously differentiable and, for all $t \ge 0$,

$$
Q_A(v,v)(t) \equiv (d/dt)\big(v(t), \mathcal{A}(t)v(t)\big) - 2\big(dv(t)/dt, \mathcal{A}(t)v(t)\big) \le 0.
$$

33. To be consistent with the notation used in (4.27).

Assume also that

(P-I) $\mathcal{P}: D_p \to H$, is a symmetric operator with $D \subseteq D_P \subseteq H$.
(P-II) $(x, \mathcal{P}x) > 0$ for all $x \in D_P$, $x \neq 0$.
(F-I) $\mathcal{F}: D \to H$ is continuously differentiable, with bounded symmetric Fréchet derivative[34] $\mathcal{F}_x$ linear operator on H and $x \to \mathcal{F}_x$ a strongly continuous map from D into $\mathcal{L}(H)$.
(F-II) Let $\mathcal{G}(x) \equiv \int_0^1 (\mathcal{F}(\rho x), x)\, d\rho$ denote the potential associated with $\mathcal{F}$,

and that the following conditions are valid:

(C-I) $(x, \mathcal{F}(x)) \geq 2(2a+1)\mathcal{G}(x)$ for $a > 0$, and all $x \in D$,
(C-II) $\mathcal{G}\big(v(t)\big) - \mathcal{G}\big(v(0)\big) = \int_0^t (\mathcal{F}\big(v(\eta)\big), v_\eta(\eta))\, d\eta$.

Let $u: [0, T) \to H$ be a solution to this problem, and assume that the preceding hypotheses hold. Then, each of the following statements are valid:
(a) If

$$\beta_0 \equiv 2\{\mathcal{G}(u_0) - 1/2\,[(u_0, \mathcal{A}(0)u_0) + (v_0, \mathcal{P}v_0)]\} > 0,$$

then the solution exists only on $[0, T)$ and $\lim_{t \to T_-} (u(t), \mathcal{P}u(t)) = +\infty$, where

$$T \leq T_{\beta_0} \equiv a^{-1}\big\{[\beta_0(u_0, \mathcal{P}u_0) + (u_0, \mathcal{P}v_0)^2]^{1/2} + (u_0, \mathcal{P}v_0)\big\}^{-1}(u_0, \mathcal{P}u_0).$$

(b) If

$$\mathcal{G}(u_0) = 1/2\,[(u_0, \mathcal{A}(0)u_0) + (v_0, \mathcal{P}v_0)],$$

$$(u_0, \mathcal{P}v_0)/(u_0, \mathcal{P}u_0) = \lambda > 0,$$

then the solution exists only on $[0, T)$ and $\lim_{t \to T_-} (u(t), \mathcal{P}u(t)) = +\infty$, where

$$T \leq (2a\lambda)^{-1}. \quad \blacksquare$$

According to the proof of Theorem 5.1 the idea of the *concavity method* is based on the construction of some positive smooth functional $\mathcal{Z}(t)$ -defined in terms of the local solution of the problem- which satisfies the following inequality

$$\mathcal{Z}''(t)\mathcal{Z}(t) - (1+a)[\mathcal{Z}'(t)]^2 \geq 0, \tag{5.1}$$

for $t > 0$, $a > 0$, $\mathcal{Z}(0) > 0$ and $\mathcal{Z}'(0) > 0$. Then $\mathcal{Z}(t) \to \infty$ for a finite time T^*. Equivalently we can select $\mathcal{Z}(t)$, such that $z(t) \coloneqq \mathcal{Z}^{1-\gamma}(t)$ for $\gamma > 1$ to be a *concave function, i.e.,*

$$z''(t) \leq 0, \;\; t > 0 \quad \text{and} \quad z(0) > 0, z'(0) < 0. \tag{5.2}$$

Then $\exists\, T^*$, $0 < T^* < \infty$ such that $z(t) \to 0$ as $t \to T^*$, $t < T^*$.

34. See in [4], [11].

From (4.2), (4.10) we define[35] (see in [10], pp. 62) the functional $\mathcal{Z}(t)$ as

$$\mathcal{Z}(t) := \|u(t)\|_{L_g^2}^2 + \delta\left\{\int_0^t \|u(s)\|_{L_g^2}^2\, ds + (T_0 - t)\|u_0\|_{L_g^2}^2\right\} + r(t+\tau)^2, \quad (5.3)$$

where $t \in [0, T_0]$ and T_0, r, τ are positive constants, to be specified latter. Since every term in the above definition is positive, we have that $\mathcal{Z}(t) > 0$,

$$\mathcal{Z}'(t) = \frac{d}{dt}\left(\|u(t)\|_{L_g^2}^2 + \delta\left\{\int_0^t \|u(s)\|_{L_g^2}^2\, ds + (T_0 - t)\|u_0\|_{L_g^2}^2\right\} + r(t+\tau)^2\right)$$

$$= 2\big(u(t), u_t(t)\big)_{L_g^2} + \delta\|u(t)\|_{L_g^2}^2 - \delta\|u_0\|_{L_g^2}^2 + 2r(t+\tau)$$

$$= 2\left\{\big(u(t), u_t(t)\big)_{L_g^2} + \delta\int_0^t \big(u(s), u_t(s)\big)_{L_g^2}\, ds + r(t+\tau)\right\}, \quad (5.4)$$

where we have used the relation below

$$2\delta\int_0^t \big(u(s), u_t(s)\big)_{L_g^2}\, ds = \delta\int_0^t \frac{d}{dt}\|u(s)\|_{L_g^2}^2\, ds = \delta\|u(t)\|_{L_g^2}^2 - \delta\|u_0\|_{L_g^2}^2,$$

and

$$\mathcal{Z}''(t) = 2\frac{d}{dt}\left(\big(u(t), u_t(t)\big)_{L_g^2} + \delta\int_0^t \big(u(s), u_t(s)\big)_{L_g^2}\, ds + r(t+\tau)\right)$$

$$= 2\frac{d}{dt}\big(u(t), u_t(t)\big)_{L_g^2} + 2\delta\frac{d}{dt}\int_0^t \big(u(s), u_t(s)\big)_{L_g^2}\, ds + 2\frac{d}{dt}r(t+\tau)$$

$$= 2\big(u_t(t), u_t(t)\big)_{L_g^2} + 2\big(u(t), u_{tt}(t)\big)_{L_g^2} + 2\delta\big(u(t), u_t(t)\big)_{L_g^2} + 2r$$

$$= 2\left\{\big(u(t), u_{tt}(t)\big)_{L_g^2} + \|u_t(t)\|_{L_g^2}^2 + \delta\big(u(t), u_t(t)\big)_{L_g^2} + r\right\}. \quad (5.5)$$

If u is a solution of (1.1), -with $f(u) = |u|^a u$- then multiplying (1.1) by gu and integrating over $\mathbb{R}^N$, we have that

35. We also recommend the papers (a) "*Non-existence of Global Solutions to Nonlinear Wave Equations with Positive Initial Energy*" by Bilgesu A. Bilgin and Varga K. Kalantarov; Communications on Pure and Applied Analysis, Vol. 17, No. 3, May 2018, pp. 987–999 and (b) "*Blow-up of Solutions to Ordinary Differential Equations arising in Nonlinear Dispersive Problems*" by Milena Dimova, Natalia Kolkovska, Nikolai Kutev; Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 68, pp. 1–16. ISSN: 1072-6691.

$$\int_{\mathbb{R}^N} g u u_{tt} dx - \|\nabla u(t)\|^2 \int_{\mathbb{R}^N} u\,\Delta u dx + \delta \int_{\mathbb{R}^N} g u u_t dx = \int_{\mathbb{R}^N} g|u|^{a+2} dx,$$

and after using some derivative arguments we derive

$$(u(t), u_{tt}(t))_{L_g^2} = -\|\nabla u(t)\|^4 - \delta \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L_g^2}^2 + \|u(t)\|_{L_g^{a+2}}^{a+2}. \tag{5.6}$$

Therefore, combining relations (5.5) and (5.6) we obtain that

$$\mathcal{Z}''(t) = 2\left\{-\|\nabla u(t)\|^4 + \|u(t)\|_{L_g^{a+2}}^{a+2} + \|u_t(t)\|_{L_g^2}^2 + r\right\}. \tag{5.7}$$

To continue with the proof of the *concavity character* we define a new functional as follows:

$$\mathcal{V}(t) := \left\{\|u(t)\|_{L_g^2}^2 + \delta \int_0^t \|u(s)\|_{L_g^2}^2 ds + r(t+\tau)^2\right\}$$
$$\times \left\{\|u_t(t)\|_{L_g^2}^2 + \delta \int_0^t \|u_t(s)\|_{L_g^2}^2 ds + r\right\} - \left\{\big(u(t), u_t(t)\big)_{L_g^2}\right.$$
$$\left. + \delta \int_0^t \big(u(s), u_t(s)\big)_{L_g^2} ds + r(t+\tau)\right\}^2. \tag{5.8}$$

We observe that $\mathcal{V}(t) \geq 0$ and from relation (5.4), (5.8) becomes

$$\mathcal{V}(t) = \left\{\mathcal{Z}(t) - \delta(T_0 - t)\|u_0\|_{L_g^2}^2\right\}$$
$$\times \left\{\|u_t(t)\|_{L_g^2}^2 + \delta \int_0^t \|u_t(s)\|_{L_g^2}^2 ds + r\right\} - \frac{1}{4}\mathcal{Z}'(t)^2,$$

or

$$\mathcal{Z}'(t)^2 = 4\left[\left\{\mathcal{Z}(t) - \delta(T_0 - t)\|u_0\|_{L_g^2}^2\right\}\right.$$
$$\left. \times \left\{\|u_t(t)\|_{L_g^2}^2 + \delta \int_0^t \|u_t(s)\|_{L_g^2}^2 ds + r\right\} - \mathcal{V}(t)\right]. \tag{5.9}$$

Hence, from equation (5.9) we get

$$\mathcal{Z}(t)\mathcal{Z}''(t) - \left(\frac{a}{4} + 1\right)\mathcal{Z}'(t)^2 \geq \mathcal{Z}(t)[\mathcal{Z}''(t) - (a+4)$$
$$\left. \times \left\{\|u_t(t)\|_{L_g^2}^2 + \delta \int_0^t \|u_t(s)\|_{L_g^2}^2 ds + r\right\}\right]. \tag{5.10}$$

Since $\mathcal{Z}(t) > 0$, we have to show the positiveness of the term inside the brackets. For this we define

$$\mathcal{H}(t) \coloneqq \mathcal{Z}''(t) - (a+4) \times \left\{ \|u_t(t)\|_{L^2_g}^2 + \delta \int_0^t \|u_t(s)\|_{L^2_g}^2 \, ds + r \right\}. \quad (5.11)$$

From relations (4.10) and (5.7), we observe that

$$\mathcal{H}(t) = 2\left\{ -\|\nabla u(t)\|^4 + \|u(t)\|_{L_g^{a+2}}^{a+2} + \|u_t(t)\|_{L^2_g}^2 + r \right\}$$

$$-(a+4) \times \left\{ \|u_t(t)\|_{L^2_g}^2 + \delta \int_0^t \|u_t(s)\|_{L^2_g}^2 \, ds + r \right\}$$

$$\geq -(a+2) \times \left\{ \|u_t(t)\|_{L^2_g}^2 + E(0) - E(t) + \frac{2}{a+2} \|u(t)\|_{L_g^{a+2}}^{a+2} + r \right\} - 2\|\nabla u(t)\|^4$$

$$= -(a+2)\{E(0) + r\} + \frac{a-2}{2} \|\nabla u(t)\|^4, \quad (5.12)$$

where we have used the relation,

$$E(t) = \|u_t(t)\|_{L^2_g}^2 + \frac{1}{2} \|u(t)\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2} \|u(t)\|_{L_g^{a+2}}^{a+2}$$

in the last equality (see Lemma 4.4 and relation (4.2)).
Fixing $r = -E(0) > 0$, inequality (5.12) becomes

$$\mathcal{H}(t) \geq \frac{a-2}{2} \|\nabla u(t)\|^4 \coloneqq \mathcal{Q}(t). \quad (5.13)$$

Then, from relations (5.10) and (5.13), we derive

$$\mathcal{Z}(t)\mathcal{Z}''(t) - \left(\frac{a}{4} + 1\right) \mathcal{Z}'(t)^2 \geq \mathcal{Z}(t)\, \mathcal{Q}(t) \geq 0, \quad (5.14)$$

which implies the concavity character of the functional $\mathcal{Z}(t)$, *i.e.*,

$$\left(\mathcal{Z}(t)^{-\frac{a}{4}}\right)'' \coloneqq -\frac{a}{4} \mathcal{Z}(t)^{-\frac{a}{4}-2} \left\{ \mathcal{Z}(t)\mathcal{Z}''(t) - \left(\frac{a}{4} + 1\right) \mathcal{Z}'(t)^2 \right\} \leq 0. \quad (5.15)$$

Having set the framework of the concavity argument we are able to state and prove the blow-up result.

Theorem 5.2. Suppose that $a \geq 2$, $N \geq 3$ and the initial energy $E(u_0, u_1)$ is negative. Then there exists a time T, where

$$0 < T \leq a^{-2}\left(-E(u_0, u_1)\right)^{-1} [\{(2\delta \|u_0\|_{L^2_g}^2 - a(u_0, u_1)_{L^2_g})^2$$

$$+ a^2(-E(u_0, u_1))\, \|u_0\|_{L^2_g}^2\}^{1/2} + 2\delta \|u_0\|_{L^2_g}^2 - a(u_0, u_1)_{L^2_g}], \quad (5.16)$$

such that the (*unique*) solution of the problem (1.1), (1.2) blows-up at time T, *i.e.*,

$$\lim_{t\to T_-} \|u(t)\|_{L_g^2}^2 = \infty. \tag{5.17}$$

Proof. We choose T_0 such that

$$\frac{4\mathcal{Z}(0)}{a\mathcal{Z}'(0)} \leq T_0. \tag{5.18}$$

We observe that $\mathcal{Z}(0) = (1+\delta T_0)\|u_0\|_{L_g^2}^2 + r\tau^2 > 0$ and from (5.3), (5.4) we have that $\mathcal{Z}'(0) = 2\left\{(u_0,u_1)_{L_g^2} + r\tau\right\} > 0$, for sufficiently large τ. Thus, for all t for which $u(t)$ exists, $\mathcal{Z}^{-a/4}(t) \leq \mathcal{Z}^{-a/4}(0) - (a/4)\, t\mathcal{Z}'(0)\mathcal{Z}^{-a/4-1}(0)$, since the graph of a concave function must lie below any tangent line of it. Hence

$$\mathcal{Z}(t) \geq \left\{\frac{4\mathcal{Z}^{a/4+1}(0)}{4\mathcal{Z}(0) - a\mathcal{Z}'(0)t}\right\}^{4/a}, \tag{5.19}$$

and therefore as $t \to T(\leq 4\mathcal{Z}(0)/a\mathcal{Z}'(0))$ from below (if $\mathcal{Z}'(0) > 0$), we note that $\mathcal{Z}(t) \to +\infty$. This is the crux of the concavity argument (see pp. 6 in [3]).
Consequently, there exists some $T \in (0, T_0]$, such that

$$\lim_{t\to T_-}\left\{\|u(t)\|_{L_g^2}^2 + \delta\int_0^t \|u(s)\|_{L_g^2}^2\, ds\right\} = \infty, \;\; i.e.\;\; \lim_{t\to T_-}\|u(t)\|_{L_g^2}^2 = \infty,$$

which proves relation (5.17).
Finally, to determine the upper bound for the blow-up time we use the relations (5.3), (5.4), (for $t=0$) and the inequality (5.18). From this we obtain (setting $r = -E(0)$)

$$\frac{4\left[(1+\delta T_0)\|u_0\|_{L_g^2}^2 + (-E(0))\tau^2\right]}{2a\left[(u_0,u_1)_{L_g^2} + (-E(0))\tau\right]} \leq T_0,$$

or

$$T(\tau) \equiv \frac{2\left[\|u_0\|_{L_g^2}^2 + \big(-E(0)\big)\tau^2\right]}{a\left[(u_0,u_1)_{L_g^2} + \big(-E(0)\big)\tau\right] - 2\delta\|u_0\|_{L_g^2}^2} \leq T_0. \tag{5.20}$$

The proper value τ_0 of τ for the blow-up, corresponds to the minimum value of $T(\tau)$.
Since

$$T'(\tau) \coloneqq \left\{\frac{2\left[\|u_0\|_{L_g^2}^2 + \big(-E(0)\big)\tau^2\right]}{a\left[(u_0,u_1)_{L_g^2} + \big(-E(0)\big)\tau\right] - 2\delta\|u_0\|_{L_g^2}^2}\right\}' \overset{\text{def}}{=} \frac{\mathcal{M}'(\tau)\mathcal{N}(\tau) - \mathcal{M}(\tau)\mathcal{N}'(\tau)}{\mathcal{N}(\tau)^2},$$

where

$$\mathcal{M}(\tau) := 2\left[\|u_0\|^2_{L^2_g} + \left(-E(0)\right)\tau^2\right],$$
$$\mathcal{N}(\tau) := a\left[(u_0,u_1)_{L^2_g} + \left(-E(0)\right)\tau\right] - 2\delta\|u_0\|^2_{L^2_g},$$

we have that

$$T'(\tau) = \frac{4E(0)\tau\left[2\delta\|u_0\|^2_{L^2_g} - a(u_0,u_1)_{L^2_g}\right] + 2aE(0)\|u_0\|^2_{L^2_g} + 2aE^2(0)\tau^2}{\left\{a\left[(u_0,u_1)_{L^2_g} + \left(-E(0)\right)\tau\right] - 2\delta\|u_0\|^2_{L^2_g}\right\}^2}. \quad (5.21)$$

Setting $T'(\tau) = 0$, we derive that $T(\tau)$ takes the minimum value on the interval $(0,\infty)$ at the value $\tau = \tau_0$, where

$$\tau_0 \equiv a^{-2}(-E(0))^{-1}\left[\left\{\left(2\delta\|u_0\|^2_{L^2_g} - a(u_0,u_1)_{L^2_g}\right)^2 + a^2(-E(0))\,\|u_0\|^2_{L^2_g}\right\}^{1/2} + 2\delta\|u_0\|^2_{L^2_g} - a(u_0,u_1)_{L^2_g}\right].$$

Q.E.D.

Remark 5.3. In the case where $a = 2$, we have the following observations:

(i)
$$\begin{aligned}\mathcal{H}(t) &\geq -(a+2)\{E(0)+r\} + (a-2)\,\|\nabla u(t)\|^4/2 \\ &= -\{E(0)+r\} - \frac{2-a}{2(a+2)}\|\nabla u(t)\|^4 \\ &\geq -\{E(0)+r\} - \frac{2-a}{(a+2)}E(0) \\ &= -\left\{\frac{4E(0)}{a+2} + r\right\} \underset{(a=2)}{=} -\{E(0)+r\},\end{aligned}$$

where we have used (3.27) in the third line.
Setting

$$r := \frac{1}{2}\|u_t(t)\|^2_{L^2_g} + \delta\int_0^t \|u_t(s)\|^2_{L^2_g}\,ds > 0,$$

and using inequality (4.19), we observe that $\mathcal{H}(t) > 0$.

(ii) The concavity character (5.15) takes the following form

$$\left(\mathcal{Z}(t)^{-1/2}\right)'' := -1/2\,\mathcal{Z}(t)^{-5/2}\{\mathcal{Z}(t)\mathcal{Z}''(t) - 3/2\,\mathcal{Z}'(t)^2\} \leq 0,$$

and by (5.18) for

$$\begin{cases} \mathcal{Z}(0) = \|u_0\|^2_{L^2_g} + \delta T_0\|u_0\|^2_{L^2_g} + 1/2\,\|u_1\|^2_{L^2_g}\tau^2 \\ \mathcal{Z}'(0) = 2\{(u_0,u_1)_{L^2_g} + 1/2\,\|u_t(t)\|^2_{L^2_g}\tau\}, \end{cases}$$

inequality (5.20) becomes

$$T(\tau) \equiv \frac{2\|u_0\|_{L_g^2}^2 + \|u_1\|_{L_g^2}^2 \tau^2}{2\left[(u_0, u_1)_{L_g^2} - \delta\|u_0\|_{L_g^2}^2\right] + \|u_1\|_{L_g^2}^2 \tau} \leq T_0.$$

From this we derive that

$$T'(\tau) = \frac{\|u_1\|_{L_g^2}^4 \tau^2 + 4(u_0, u_1)_{L_g^2}\|u_1\|_{L_g^2}^2 \tau - 2\|u_0\|_{L_g^2}^2 \|u_1\|_{L_g^2}^2 [1 + 2\delta\tau]}{\left\{2\left[(u_0, u_1)_{L_g^2} - \delta\|u_0\|_{L_g^2}^2\right] + \|u_1\|_{L_g^2}^2 \tau\right\}^2},$$

and that $T(\tau)$ takes the minimum value on the interval $(0, \infty)$ at the value $\tau = \tau_0'$, where

$$\tau_0' \coloneqq a^{-1}\left\{2\left[\delta\|u_0\|_{L_g^2}^2 - (u_0, u_1)_{L_g^2}\right] + \sqrt{2} \times \left[\left(\delta\|u_0\|_{L_g^2}^2 - (u_0, u_1)_{L_g^2}\right)^2 + \alpha/2 \, \|u_0\|_{L_g^2}^2\right]^{1/2}\right\}.$$

From Theorem 5.2 and Remark 5.3 we conclude that as $t \to T_-$ the solution $u \coloneqq u(x,t)$ of the problem (1.1)-(1.2), blows-up, *i.e.*, the natural system described by this mathematical model goes through a transitional change[36]. For instance, in the case where the solution represents the transverse displacement of a string, the blow-up corresponds to the breaking of the string.

36. See "*Contemporary Issues in Systems Science and Engineering*" by MengChu Zhou, Han-Xiong Li, Margot Weijnen, Wiley-IEEE Press, April 2015, ISBN: 978-1-118-27186-5; pp. 82.

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